Lecture 1. Basic concepts of statistical optics.

The subject of quantum optics and its relation to other subjects. Links to statistical optics, nonlinear optics, quantum information. Applications: quantum information, metrology.
Basic notions of statistical optics. Random signal, stationary and ergodic processes. Analytic signal, spectral density, correlation functions. Wiener-Khinchin theorem.

1. *Introduction to quantum optics.*

HBT experiment. Traditionally, it is assumed that quantum optics started from the Hanbury Brown-Twiss (HBT) experiment (1956). In this experiment, or rather a series of experiments, Robert Hanbury Brown and Richard Twiss observed intensity correlations for the radiation of a mercury lamp and some bright stars. After a beamsplitter (in the case of a mercury lamp) or at two spatially separated (but not too much) points the intensities measured by two detectors were fluctuating, and these fluctuations were correlated. These experiments were immediately explained in terms of photons ('light consists of photons'): it looked like in the radiation of thermal sources, such as stars and gas discharge lamps, photons 'bunch'. In fact, these experiments have perfectly classical explanation, without any photons: one has just to know that the intensity has fluctuations in time, with a certain probability distribution. In particular, for thermal sources (and most sources are thermal), the distribution is negative-exponential,

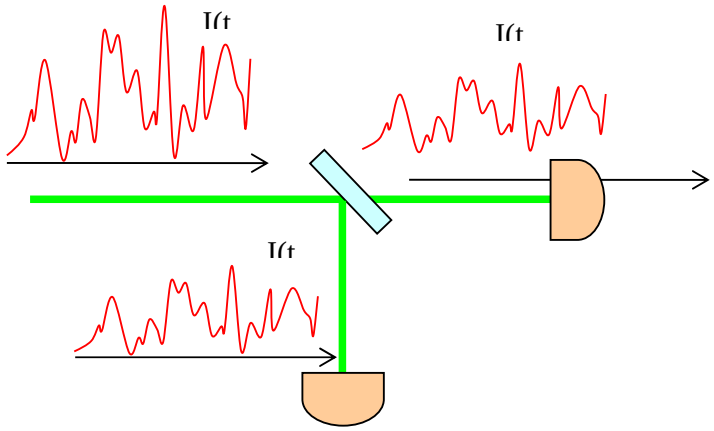

Fig.1

$P(I) = \frac{1}{\langle I \rangle} e^{-I/\langle I \rangle}$. (We will derive it later.)

This fully explains the results of HBT experiments.

Photons. Real quantum optics started when so-called nonclassical light sources were discovered. Nonclassical light (which will be considered in detail further) requires quantum theory for its description, and the first examples of such light were single-photon and two-photon states of light (single photons and photon pairs). The latter were discovered first, in 1967 (photon pairs created via cascaded atom decay and parametric down conversion).

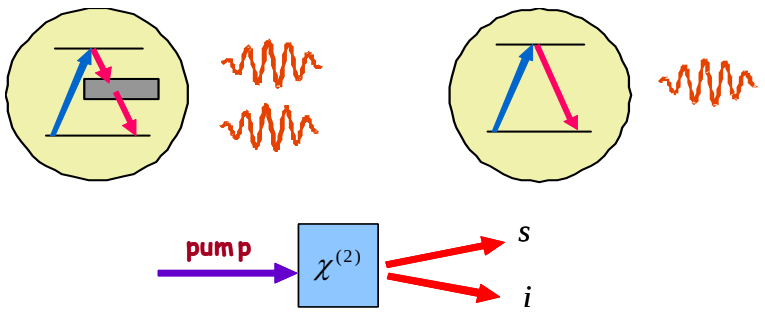

Fig.2

The former were obtained only in 1977, in experiments on the fluorescence of single atoms. For the description of such sources, quantum theory is needed, and this is what quantum optics (QO) is about: describing nonclassical behavior of light.

Nonclassical effects. Just a short list of these nonclassical features:

- Anti-bunching: no correlations, or, rather, anti-correlations in the HBT setup;
- The existence of shot noise (photon noise) in the detection of light and, which is even stranger, the possibility to suppress it (which proves that it is not caused by detection!)
- Negative values of what seems to be a probability (Wigner function)
- Bell's inequalities violation: a proof that a single quantum particle (a photon) behaves probabilistically and in the general case has no *a priori* properties like polarization, wavelength, wavevector.

They will be discussed in the section about nonclassical light.

Links to other fields. We have already come across three subjects that are very close to quantum optics: statistical optics, nonlinear optics and atom spectroscopy (or at least interaction between atom and light). Indeed, quantum optics is closely linked to these 3 subjects. According to this, the first few lectures will be focused on statistical optics, and then one lecture will be specially devoted to the interaction between light and atoms and one, to the fundamentals of nonlinear optics. It is assumed that you already know classical optics, theory of electromagnetic radiation, and quantum mechanics.

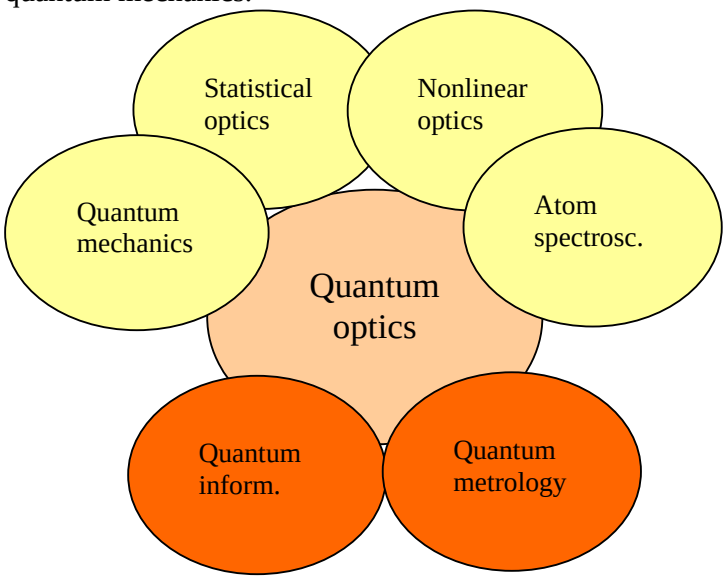

Fig.3

Quantum information. Probably, the most important modern use of quantum optics is in quantum information. The idea of quantum information is replacing a usual bit (0 or 1) by a qubit, a superposition of two quantum states, one representing 0 and the other 1: $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$. It turns out that the possibility to use superposition (a typically quantum thing) offers certain advantages. These are: faster solution of a certain class of computational problems and the fragility of this type of information, which, in its turn, provides secrecy. Secret transmission of quantum information, namely quantum key distribution (QKD), is the best-developed application of quantum information. Photons or other states of light are of course not the only possible carriers for quantum information but they are remarkably good for the information transmission because they interact very little. (This very fact makes them inconvenient for processing the information.)

Polarization. Because the simplest way to encode information is binary encoding, if photons are to be used for this purpose, it is the polarization that immediately comes into mind. Polarization is described in 2D space: roughly, horizontal or vertical, right or left circular. With more complicated states of light rather than single photons, higher-dimensional encoding can be also achieved with polarization. For this reason, one lecture will be devoted to polarization.

Applications. It is not fair to omit the applications of quantum optics. One I have already mentioned: QKD. It is now at a commercial level: companies sell devices for QKD. The next important one, probably, is quantum metrology, which is developed to the level of state standards in national metrology institutes and usage in gravitational-wave antennas. Less developed one is quantum imaging. So far, no separate lecture is planned but this can be decided later.

2. *Basic notions of statistical optics.*

We will start with classical statistical optics. For this start, randomness will be understood as classical randomness – further we will discuss what really quantum randomness is.

Random field.
A random field is a particular case of a random process (an extension of the notion of a random number) $u(t)$. It can be characterized by probabilities $p(u,t)$ (first-order probability) or joint probabilities $p(u_1,u_2;t_1,t_2)$, where
$u_1 = u(t_1), \quad u_2 = u(t_2),$ or $p(u_1,u_2,...u_k,t_1,t_2,...t_k).$
The mean value is then given by
$\langle u \rangle = \int du p(u,t)u(t)$ over the realizations $u^{(1)}(t)$, $u^{(2)}(t)$,
$u^{(3)}(t)$ … (shown by colour).

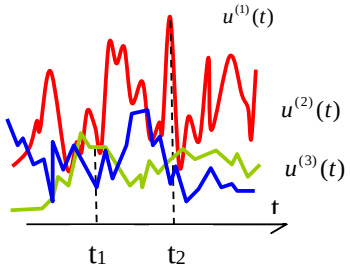

Fig.4

In some (most) books on statistics, there is a distinction between ensemble averaging, and time averaging. Further we will also introduce quantum averaging. Not to write three different symbols (usually E[u], $\bar{u}$ <u>), we will write brackets everywhere. Because further we will consider only quantum averaging.
One can also define the moments
$\langle u^k \rangle = \int du p(u,t)u^k(t)$
or the joint moments
$\langle u_1{}^k u_2{}^p \rangle = \int du p(u_1,u_2;t_1,t_2)u_1{}^k(t_1)u_2{}^p(t_2)$

Stationarity. There are two definitions.
In the narrow sense (strictly), a stationary process should have the most joint probability distribution time-independent:
$p(u_1,u_2,...u_k;t_1,t_2,...t_k) = p(u_1,u_2,...u_k;t_1-T,t_2-T,...t_k-T).$
But this is often not necessary, so there is a definition of wide-sense stationarity, or simply stationarity:
(1) $\langle u \rangle$ does not depend on time;
(2) $\langle u_1 u_2 \rangle \equiv \int du p(u_1,u_2;t_1,t_2)u_1(t_1)u_2(t_2)$ depends only on $\tau \equiv t_1 - t_2$.
(Further, I will always denote definition by the ‘≡’ sign; sometimes it is denoted by ‘=:’, as in computer science. But it is important to distinguish between a definition and a result of calculation.)
In other words, wide-sense stationarity only implies that the mean value and second-order moment (correlation function) are time-independent.
Strict stationarity always leads to wide-sense stationarity, but not vice versa.

Spatial random distributions. There is full analogy between space and time; so similarly let us define spatial random functions and, by analogy to stationarity, we will define homogeneity. Also, in the narrow sense (strictly) and in the broad sense.

Ergodicity. An ergodic process is one for which the probability distribution along the time axis ('horizontally' in Fig.4) is the same as over different realizations ('vertically' in Fig. 4). Then averaging over time (simpler and operational) provides the same information as averaging over realizations. There is no special term for a process that can be averaged over space, but we will keep in mind that there is a close analogy.

Examples.

The relation between ergodicity, strict stationarity, and wide-sense stationarity (simply stationarity) is shown in Fig.5.

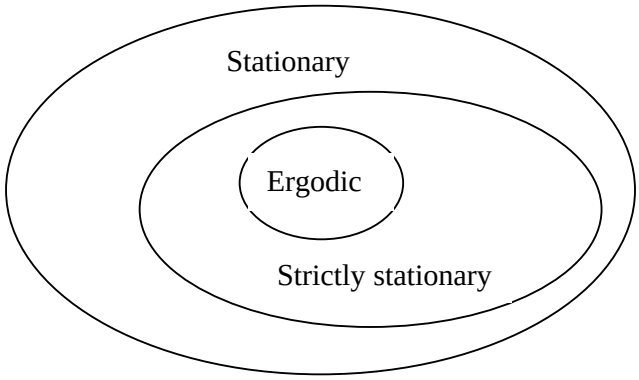

Fig.5

Random electric field $E(t)$ is just a particular case of a random process. It is of course a real function, characterized, for instance, by random amplitude and random phase: $\mathrm{E}(t) = A(t)\cos(\omega t + \Phi(t))$.

Analytic signal. It is very convenient to describe this electric field by a complex signal, so that the observed field is its real part. To introduce this *analytic signal*, let us decompose the field into a Fourier integral

$$\mathrm{E}(t) = \int_{-\infty}^{\infty} d\omega e^{-i\omega t} E(\omega)$$

and separate the integral in two parts:

$\mathrm{E}^{(-)}(t) \equiv \int_{-\infty}^{0} d\omega e^{-i\omega t} E(\omega)$ and $\mathrm{E}^{(+)}(t) \equiv \int_{0}^{\infty} d\omega e^{-i\omega t} E(\omega)$. These are negative-frequency and positive-frequency fields.

They are complex conjugates of each other as

$$(\mathrm{E}^{(+)}(t))^* = \int_{0}^{\infty} d\omega e^{i\omega t} E^*(\omega) =$$

$$= \int_{-\infty}^{0} d\omega e^{-i\omega t} E^*(-\omega) =$$

$$= \int_{-\infty}^{0} d\omega e^{-i\omega t} E(\omega) = \mathrm{E}^{(-)}(t),$$

because $E(\omega)$ is the Fourier-transform of a real function and therefore has the property $E^*(\omega) = E(-\omega)$.

The positive-frequency field $\mathrm{E}^{(+)}(t)$ will be called the analytic signal, and the real field is

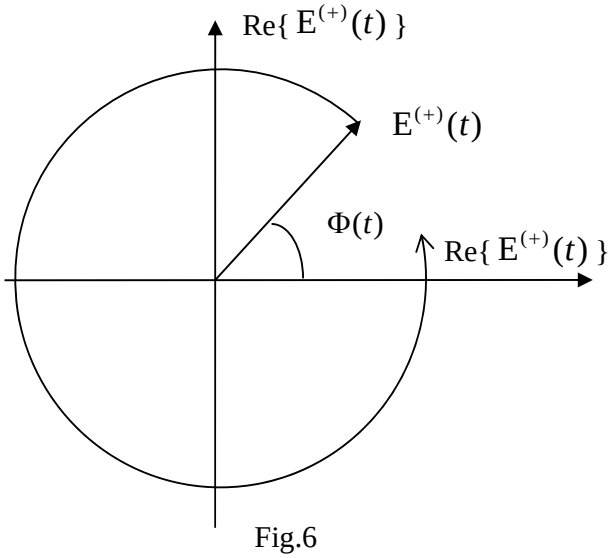

Fig.6

related to it as $E(t) = \mathrm{E}^{(+)}(t) + \mathrm{E}^{(-)}(t) = 2\,\mathrm{Re}\{\mathrm{E}^{(+)}(t)\}$.
Figure 6 shows the picture for a monochromatic field. The arrow depicting the analytic signal is rotating with the light frequency. If the amplitude changes in time, the trajectory differs from a circle. This is the case for a random field. Both the amplitude and the phase can be random.

Intensity is proportional to the squared modulus of the analytic signal; in reality there is a dimensional coefficient but we will omit it and write simply
$I(t) = \mathrm{E}^{(+)}(t)\mathrm{E}^{(-)}(t) = |\mathrm{E}^{(+)}(t)|^2$.
This is instantaneous intensity, another random process. It can be measured by various types of detectors: single-photon ('click') detectors, p-i-n diodes, CCD cameras, bolometers, etc. All these detectors are sensitive to the intensity, not to the field.
We will further discuss its mean value $\langle I(t) \rangle$ and higher moments. For instance, variance
$\Delta I^2 \equiv \langle I^2(t) \rangle - \langle I(t) \rangle^2$.

Spectral density. We also define the spectral density as
$S(\omega) \equiv \left\langle |E(\omega)|^2 \right\rangle$. This definition is not strict from the very beginning [when we define $E(\omega)$], because, in the general case, a random process $E(t)$ may not have a Fourier transform. But in reality light starts at some point (a Ph.D. student turns on the laser) and stops at some point (he turns off the laser and goes home), so everything is finite, integrable etc.

Measurement of the spectrum (spectral density). By definition, the spectrum of positive-frequency field contains only positive frequencies. It can be measured by an optical spectrometer, and if the central frequency is ω_0, so that the analytic signal is $\mathrm{E}^{(+)}(t) = E_0(t)\exp\{-i\omega_0 t\}$, then the spectrum looks as shown in Fig.7. Typical widths are: kHz for a good cw laser and 10 THz for a femtosecond laser or a laser pointer.

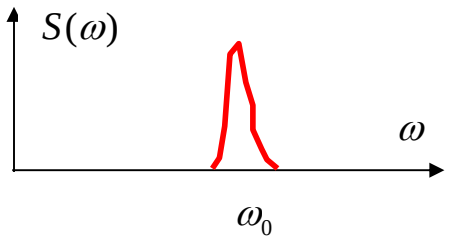

Fig.7

Filtering. If a filter is used, such as, for instance, an interference filter, or a monochromator slit, or a Fabry-Perot interferometer, then the resulting spectral density is obtained by multiplying the transmission of the filter and the initial spectral density:
$S(\omega) \to S(\omega)T(\omega)$.

First-order correlation functions. This terminology ('first-order') is consistent with some books (Glauber, Klyshko) but inconsistent with others (Mandel&Wolf). We will use it in this course. It means that the correlation function (CF) is of the first order in the intensity, second-order in the field.
The definition is
$G^{(1)}(t_1, t_2, r_1, r_2) \equiv \left\langle \mathrm{E}^{(-)}(t_1, r_1)\mathrm{E}^{(+)}(t_2, r_2) \right\rangle$.

For a stationary (even simply stationary) and homogeneous light field, $G^{(1)}(t_1,t_2,r_1,r_2)=G^{(1)}(t_2-t_1,r_2-r_1)\equiv G^{(1)}(\tau,\rho)$, the CF depends only on the relative time and displacement. It has some other interesting properties:

(1) $G^{(1)}(0)\geq 0$ - in fact, $G^{(1)}(0)\equiv\langle I(t)\rangle$.

(2) $G^{(1)}(\tau)=G^{(1)}*(-\tau)$, and similarly for the spatial CF. This follows from the stationarity and the definition:

$$G^{(1)}(\tau)\equiv\left\langle \mathrm{E}^{(-)}(t)\,\mathrm{E}^{(+)}(t+\tau)\right\rangle=\left\langle \mathrm{E}^{(-)}(t-\tau)\,\mathrm{E}^{(+)}(t)\right\rangle=\left\langle \mathrm{E}^{(+)}(t)\,\mathrm{E}^{(-)}(t-\tau)\right\rangle=G^{(1)}*(-\tau).$$

(3) Cauchy-Schwarz inequality:

$$\left|G^{(1)}(\tau)\right|^2=\left|\left\langle E^{(-)}(t)E^{(+)}(t+\tau)\right\rangle\right|^2\leq\left\langle E^{(-)}(t)E^{(+)}(t)\right\rangle\left\langle E^{(-)}(t+\tau)E^{(+)}(t+\tau)\right\rangle=[G^{(1)}(0)]^2,$$

Hence $\left|G^{(1)}(\tau)\right|\leq G^{(1)}(0)$. This classical inequality will be violated in quantum optics!

Wiener-Khinchin theorem is probably the most important theorem in statistical optics: the spectral density and the first-order CF are Fourier-related,

$$G^{(1)}(\tau)=\int S(\omega)\exp\{i\omega\tau\}d\omega.$$

Proof: write the relation between the field and its Fourier transform,

$$\mathrm{E}(t)=\int_{-\infty}^{\infty}d\omega e^{-i\omega t}E(\omega),$$

then build the CF,

$$G^{(1)}(\tau)\equiv\left\langle \mathrm{E}^{(-)}(t)\,\mathrm{E}^{(+)}(t+\tau)\right\rangle=\left\langle \int_{-\infty}^{0}d\omega' e^{-i\omega' t}E(\omega')\int_{0}^{\infty}d\omega e^{-i\omega(t+\tau)}E(\omega)\right\rangle.$$

Because the spectrum contains only positive frequencies, the integral can be taken from minus infinity to plus infinity. Then

$$G^{(1)}(\tau)=\int_{-\infty}^{\infty}d\omega' e^{-i\omega' t}\int_{-\infty}^{\infty}d\omega e^{-i\omega(t+\tau)}\left\langle E(\omega')E(\omega)\right\rangle=\int_{-\infty}^{\infty}d\omega' e^{-i\omega' t}\int_{-\infty}^{\infty}d\omega e^{i\omega(t+\tau)}\left\langle E(\omega')E*(\omega)\right\rangle=$$

$$=\int_{-\infty}^{\infty}d\omega' e^{-i(\omega'-\omega)t}\int_{-\infty}^{\infty}d\omega e^{i\omega\tau}\left\langle E(\omega')E*(\omega)\right\rangle.$$

Because of stationarity, this expression should not have dependence on t. Hence, the correlator should give a delta function,

$\left\langle E(\omega')E*(\omega)\right\rangle=\delta(\omega'-\omega)\left\langle E(\omega)E*(\omega)\right\rangle$, and then we obtain the theorem:

$$G^{(1)}(\tau)=\int_{-\infty}^{\infty}d\omega e^{i\omega\tau}\left\langle E(\omega)E*(\omega)\right\rangle.$$

(Here we made a lot of assumptions, probably wrong mathematically but justified from the physical viewpoint!)

The same relation is valid for spatial variables.

This is a very useful relation: there is one-to-one correspondence between the spectrum (measured by a spectral device) and the correlation function, which can be measured using an interferometer (will be discussed at the next lecture). Note that neither of the two is sensitive to the phase of the field, they both only give information about the modulus of the spectral amplitude.

Some interesting examples of spectra and CFs.

1. Light with a Gaussian spectrum will have a Gaussian CF.
2. Rectangular spectrum leads to a CF shaped as the 'sinc- function': $G^{(1)}(\tau) = \frac{\sin(K\tau)}{K\tau}$.
3. Sinc-squared-shaped spectrum leads to a triangular CF, and vice versa.
4. What is the CF for a Gaussian spectrum modulated by the Fabry-Perot transmission?

If the spectrum cannot be measured (too narrow) then the CF sometimes is easier to measure. This is called Fourier-spectroscopy. For instance, radiation of a fs laser with width 10 nm is modulated by a Fabry-Perot; the resulting spectrum consists of peaks with width 0.04 nm and the spectrometer resolution is only 0.1 nm. What to do? Measure the CF (estimate the dimensions).

It is convenient to normalize: $g^{(1)}(t_1, t_2, r_1, r_2) \equiv \frac{G^{(1)}(t_1, t_2, r_1, r_2)}{\sqrt{\langle I(t_1, r_1)\rangle \langle I(t_2, r_2)\rangle}}$

Home task: A broken diode laser emits two spectral lines, both with Gaussian shapes and the same width 0.1 nm, separated by 5 nm distance. What will be the shape of the first-order correlation function?

Books:

1. Goodman, Statistical optics, Chapters 2-3
2. Mandel & Wolf, Optical coherence and quantum optics, Section 2.2
3. Klyshko, Physical foundations of quantum electronics, Section 7.2