Lecture 10: Polarization states.

Two-photon states. Encoding qubits and qutrits. 'Hidden polarization' effect. Its classical description. Two-photon polarization entangled state.

At the last lecture, we discussed a single-photon state as the simplest quantum state from the viewpoint of polarization. At this lecture we move a bit further and introduce a two-photon state.

1. Polarized photon as a qubit.

One of the main applications of polarized single photons is to encode quantum information and to transmit it. Polarization is very little affected by atmospheric scattering (remember the experiment with looking at the sky through a polarizer); therefore a polarized photon is the best carrier of quantum information in free space. Polarized photons are used in quantum key distribution (QKD); this will be the subject of the last lecture (12). But now we will introduce the tools for QKD: single photons, their measurement, and entangled photons.

A qubit is encoded into a polarized photon as

$$|\Psi\rangle = \alpha|1\rangle_H + \beta|1\rangle_V, \quad |\alpha|^2 + |\beta|^2 = 1. \tag{1}$$

How to prepare such a photon? Single photons are emitted by single-photon emitters: single atoms, molecules, quantum dots, color centers in diamond. Ideally, it should be 'on demand' preparation, but in reality, the state prepared is a superposition or even a mixture of such a photon with the vacuum. One can also prepare a single photons from a pair of correlated photons: the detection of one of them 'heralds' the appearance of the other one.

As long as a single photon is prepared, it can be converted into any state of the form (1). For instance, initially it is linearly polarized; then, by transmitting it through a set of waveplates one can prepare any arbitrary state (1).

Let us calculate the Stokes parameters for this state.

For these and further calculations, we will need the equations describing the action of the photon creation and annihilation operators on Fock states. Namely,

$$\begin{aligned} a|N\rangle &= \sqrt{N}|N-1\rangle, \\ a^+|N\rangle &= \sqrt{N+1}|N+1\rangle. \end{aligned} \tag{2}$$

We will use them in what follows.

Also, because photon creation/annihilation operators are quantum counterparts of the classical fields, their transformations by polarization elements are the same, described by the Jones matrices:

$$\begin{aligned} \begin{pmatrix} a_H' \\ a_V' \end{pmatrix} &= D\begin{pmatrix} a_H \\ a_V \end{pmatrix}, \\ D &= \begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix}. \end{aligned} \tag{3}$$

We start with the calculation of the total photon number for state (1),

$$\langle \hat{S}_0 \rangle = \{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\} \hat{S}_0 \{\alpha|1\rangle_H + \beta|1\rangle_V\} = \{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\} (a_H^+ a_H + a_V^+ a_V) \{\alpha|1\rangle_H + \beta|1\rangle_V\} =$$

$$= \{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\} \{\alpha|1\rangle_H + \beta|1\rangle_V\} = |\alpha|^2 + |\beta|^2 = 1.$$

Of course we obtained 1 because the number of photons is 1 ☺
Now, the other Stokes parameters are

$$\langle \hat{S}_1 \rangle = \{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\} \hat{S}_1 \{\alpha |1\rangle_H + \beta |1\rangle_V\} = \{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\}(a_H^+ a_H - a_V^+ a_V)\{\alpha |1\rangle_H + \beta |1\rangle_V\} =$$
$$= \{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\}\{\alpha |1\rangle_H - \beta |1\rangle_V\} = |\alpha|^2 - |\beta|^2;$$
$$\langle \hat{S}_2 \rangle = \{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\} \hat{S}_2 \{\alpha |1\rangle_H + \beta |1\rangle_V\} = \{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\}(a_H^+ a_V + a_V^+ a_H)\{\alpha |1\rangle_H + \beta |1\rangle_V\} =$$
$$= \{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\}\{\alpha |1\rangle_V + \beta |1\rangle_H\} = \alpha^* \beta + \beta^* \alpha = 2\,\mathrm{Re}(\alpha^* \beta);$$
$$\langle \hat{S}_3 \rangle = \{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\} \hat{S}_3 \{\alpha |1\rangle_H + \beta |1\rangle_V\} = -i\{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\}(a_H^+ a_V - a_V^+ a_H)\{\alpha |1\rangle_H + \beta |1\rangle_V\} =$$
$$= -i\{\alpha^* \langle 1|_H + \beta^* \langle 1|_V\}\{-\alpha |1\rangle_V + \beta |1\rangle_H\} = -i(\alpha^* \beta - \beta^* \alpha) = 2\,\mathrm{Im}(\alpha^* \beta).$$

We obtain

$$\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 + \langle \hat{S}_3 \rangle^2 = \left(|\alpha|^2 - |\beta|^2\right)^2 + 4\,\mathrm{Re}^2(\alpha^* \beta) + 4\,\mathrm{Im}^2(\alpha^* \beta) =$$
$$= \left(|\alpha|^2 - |\beta|^2\right)^2 + 4|\alpha|^2 |\beta|^2 = \left(|\alpha|^2 + |\beta|^2\right)^2 = 1,$$

and hence the degree of polarization is

$$P = \frac{\sqrt{\langle \hat{S}_1 \rangle^2 + \langle \hat{S}_2 \rangle^2 + \langle \hat{S}_3 \rangle^2}}{\langle \hat{S}_0 \rangle} = 1.$$

The conclusion is: a single photon is always polarized.

2. Measurement of polarized photons

An important part of QKD is measurement of the polarization state of a single photon. At Lecture 9, we considered this measurement: the recipe is to project the state on the eigenstates of the Stokes operators.

We obtained that the eigenstates of $\hat{S}_1$ are $|1\rangle_H, |1\rangle_V$, the eigenstates of $\hat{S}_2$ are $|1\rangle_D, |1\rangle_A$, and the eigenstates of $\hat{S}_3$ are $|1\rangle_L, |1\rangle_R$, all with the eigenvalues 1, -1. For measuring a certain Stokes observable, one should perform the projection on the corresponding eigenstate. This is done with the setup shown in Fig.1, or the same setup with a HWP@22.5°, or the same setup with a QWP@45°.

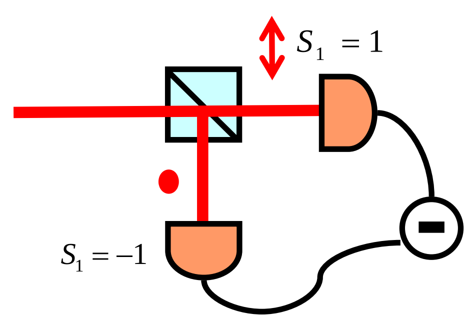

Fig. 1

Importantly, if the state is really a single photon and it is sent only once, it is impossible with a single such setup to measure all three Stokes parameters. For instance, we can measure $\hat{S}_1$ and therefore distinguish if the state is $|1\rangle_H$ or $|1\rangle_V$, but if the state is, for instance, $|1\rangle_D$, we can take it for $|1\rangle_H$ or $|1\rangle_V$ with 50% probability.

3. Two-photon state

Produced through parametric down-conversion, or four-wave mixing, or two-photon transitions in atoms etc.

We will first consider *orthogonally polarized photons*,

$$|\Psi\rangle = |1\rangle_H |1\rangle_V . \qquad (4)$$

For instance, such states can be produced through type-II spontaneous parametric down-conversion (SPDC), inverse process to the second harmonic generation of type eo->e. In type-II SPDC, a single pump photon (e-polarized) splits in two daughter photons, one of them e-polarized and the other o-polarized. The resulting state will be (4) if the geometry is as shown in Fig. 2: e corresponds to V, o corresponds to H.

The phase matching is

$$\vec{k}_e(2\omega) = \vec{k}_o(\omega) + \vec{k}_e(\omega),$$

where 2ω is the pump frequency and ω is the down-converted frequency.

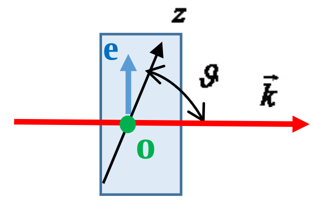

Fig. 2

Let us calculate the degree of polarization for state (4). The number of photons is

$$\langle \hat{S}_0 \rangle = \langle 1|_H \langle 1|_V \hat{S}_0 |1\rangle_H |1\rangle_V = \langle 1|_H \langle 1|_V (a_H^+ a_H + a_V^+ a_V) |1\rangle_H |1\rangle_V = \langle 1|_H \langle 1|_V (|1\rangle_H |1\rangle_V + |1\rangle_H |1\rangle_V) = 2 .$$

Of course it is 2 because we considered a two-photon state ☺ . The other Stokes parameters are

$$\langle \hat{S}_1 \rangle = \langle 1|_H \langle 1|_V \hat{S}_1 |1\rangle_H |1\rangle_V = \langle 1|_H \langle 1|_V (a_H^+ a_H - a_V^+ a_V) |1\rangle_H |1\rangle_V = \langle 1|_H \langle 1|_V (|1\rangle_H |1\rangle_V - |1\rangle_H |1\rangle_V) = 0;$$

$$\langle \hat{S}_2 \rangle = \langle 1|_H \langle 1|_V \hat{S}_2 |1\rangle_H |1\rangle_V = \langle 1|_H \langle 1|_V (a_H^+ a_V + a_V^+ a_H) |1\rangle_H |1\rangle_V = \langle 1|_H \langle 1|_V (\sqrt{2}|2\rangle_H |0\rangle_V + \sqrt{2}|0\rangle_H |2\rangle_V) = 0;$$

$$\langle \hat{S}_3 \rangle = \langle 1|_H \langle 1|_V \hat{S}_3 |1\rangle_H |1\rangle_V = -i \langle 1|_H \langle 1|_V (a_H^+ a_V - a_V^+ a_H) |1\rangle_H |1\rangle_V =$$

$$= -i \langle 1|_H \langle 1|_V (\sqrt{2}|2\rangle_H |0\rangle_V - \sqrt{2}|0\rangle_H |2\rangle_V) = 0.$$

As a result, the state is unpolarized: $P = 0$. It means that if we look at this photon pair through a polarizer, the transmitted intensity (photon number) will not depend on the orientation of this polarizer. (Question: what will be this photon number?) Or, in other words, if we put a polarization prism, the probability of the photon to pass will be the same as the probability to be reflected, regardless of the prism orientation.

4. Hidden polarization effect

However, one can still see some polarization dependence for this state – for instance, retrieve the original directions of e and o polarizations. For this, one should measure the second-order correlation function (CF),

$$G^{(2)} \equiv \langle (a^+)^2 a^2 \rangle . \qquad (5)$$

It is measured by splitting the beam on a beamsplitter and sending the two outputs to two detectors; then the rate of coincidence counts will scale as (5).

Let us calculate the CF (5) for different polarization modes. Obviously, $G_H^{(2)} = G_V^{(2)}$ because state (4) is symmetric. Using (2), we get

$$G_H^{(2)} \equiv \langle (a_H^+)^2 a_H^2 \rangle = \langle 1|_H \langle 1|_V (a_H^+)^2 a_H^2 |1\rangle_H |1\rangle_V = \langle 1|_H \langle 1|_V \sqrt{2} \cdot \sqrt{2} |1\rangle_H |1\rangle_V = 2$$

But let us now calculate this CF for diagonally polarized light. It means that in the first case, we transmit light through a polarizer that selects horizontal polarization; then measure the CF. In the second case we rotate this polarizer 45° (Fig.3).

For the 45° polariyation,

$$G_D^{(2)} \equiv \left\langle (a_D^+)^2 a_D^2 \right\rangle,$$

where $a_D = (a_H + a_V)/\sqrt{2}$. Then,

$$G_D^{(2)} = \frac{1}{4}\langle 1|_H \langle 1|_V (a_H^+ + a_V^+)^2 (a_H + a_V)^2 |1\rangle_H |1\rangle_V =$$

$$= \frac{1}{4}\langle 1|_H \langle 1|_V [(a_H^+)^2 + 2a_H^+ a_V^+ + (a_V^+)^2] \times$$

$$\times [(a_H)^2 + 2a_H a_V + (a_V)^2] |1\rangle_H |1\rangle_V =$$

$$= \frac{1}{2}\langle 1|_H \langle 1|_V [(a_H^+)^2 + 2a_H^+ a_V^+ + (a_V^+)^2] |1\rangle_H |1\rangle_V = 1.$$

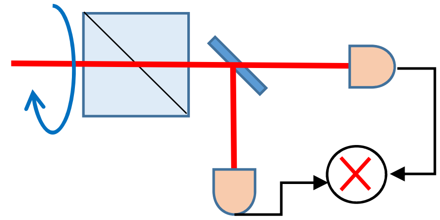

Fig. 3

It means that if we look at the state after a polarizer, the photon number will not change but the CF will.

Even more surprising will be the behavior of the cross-correlation function that can be measured for the two outputs of a polarization prism (Fig.4). If the prism is not rotated, the rate of coincidences will scale as

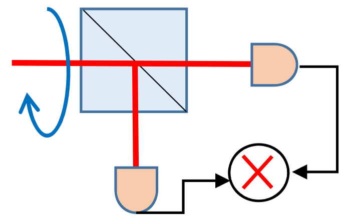

Fig. 4

$$G_{HV}^{(2)} \equiv \left\langle a_H^+ a_V^+ a_H a_V \right\rangle = \langle 1|_H \langle 1|_V a_H^+ a_V^+ a_H a_V |1\rangle_H |1\rangle_V = \langle 1|_H \langle 1|_V 1 \cdot 1 |1\rangle_H |1\rangle_V = 1.$$

But if the prism is rotated 45°, then the rate will be given by

$$G_{AD}^{(2)} \equiv \left\langle a_A^+ a_D^+ a_A a_D \right\rangle = \frac{1}{4}\langle 1|_H \langle 1|_V (a_H^+ - a_V^+)(a_H^+ + a_V^+)(a_H - a_V)(a_H + a_V) |1\rangle_H |1\rangle_V =$$

$$= \frac{1}{4}\langle 1|_H \langle 1|_V [(a_H^+)^2 - (a_V^+)^2][a_H^2 - a_V^2] |1\rangle_H |1\rangle_V = 0.$$

We have considered two extreme cases; as the polarizer (with the whole setup) rotates, the CF will change as shown in Fig.5. Therefore, while the transmitted number of photons will not depend on the prism orientation (black line), the CF will show modulation with 100% visibility (red line). This effect is called 'hidden polarization': light is unpolarized in terms of the photon number but is polarized in its second order.

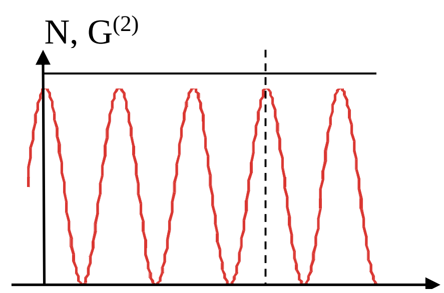

Fig. 5

5. Entangled photons.

Type-II SPDC can be also in a different geometry, namely non-collinear. The wavevectors of the two daughter photons are in this case not parallel to the pump wavevector. Calculation shows that at the degenerate frequency, o and e photons are emitted along two different cones, displaced along the optic axis (Fig.6). These cones intersect along two lines, A and B in the figure. (In the case we considered previously, these two lines become a single line.)

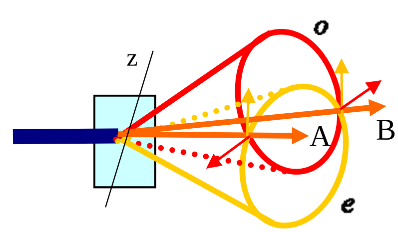

Fig. 6

Along each line, there is both e-polarized photon (V) and o-polarized photon (H). Because of the phase matching,

$$\vec{k}_p(2\omega) = \vec{k}_A(\omega) + \vec{k}_B(\omega), \qquad (6)$$

the two photons should be always emitted symmetrically, and always with orthogonal polarizations. It is therefore possible that photon A is H-polarized and photon B, V polarized, and vice versa. The state is therefore (due to the both possibilities)

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle_{AH}|1\rangle_{BV} + |1\rangle_{AV}|1\rangle_{BH}).$$

The standard way to write it is (denoting $|1\rangle_H$ as $|H\rangle$)

$$|\Psi^{(+)}\rangle = \frac{1}{\sqrt{2}}(|H\rangle_A|V\rangle_B + |V\rangle_A|H\rangle_B). \qquad (7)$$

The sign in this superposition depends on the phase shift in the crystal, and another state is possible,

$$|\Psi^{(-)}\rangle = \frac{1}{\sqrt{2}}(|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B). \qquad (7')$$

By placing a HWP in arm B, one can rotate the polarization of this photon by 90°, and obtain the states

$$|\Phi^{(\pm)}\rangle = \frac{1}{\sqrt{2}}(|H\rangle_A|H\rangle_B \pm |V\rangle_A|V\rangle_B). \qquad (8)$$

Equations (7)-(8) describe the so-called Bell states, or polarization entangled states. They demonstrate the definition of entanglement: neither of them can be written as a product state for photons 1 and 2.

Another feature of entangled states: for each of the photons A,B the state is mixed, i.e., The degree of polarization is zero. (Test this at the problem class.) But there is correlation between the polarization states: if photon A is detected with H-polarization, its match photon B has to have V-polarization.

Literature:

1. D. Bouwmeester, A. Ekert, A. Zeilinger (Eds.), The Physics of Quantum Information, sec. 3.4.
2. D N Klyshko, Physics – Uspekhi, 41 (9) 885 - 922 (1998).