## Lecture 2. The Jones vector and the Jones matrices.

The Jones vector. Jones vectors for different polarization states. Orthogonality of polarization modes. Transformations of the Jones vector: SU(2) matrices, analogy with a beamsplitter. Phase (retardation) plates and polarization rotators.

***1. The Jones vector.*** We start with the classical description of the polarization state of light. It is defined in terms of the electric field vector, which is conveniently described by the complex *analytic signal*. Here, we will take into account that it is a vector:

$$\vec{E}^{(+)}(t) = \vec{E}_0(t)e^{ikz-i\omega t} . \qquad (1)$$

The real field vector is then

$$\vec{E}(t) = \vec{E}^{(+)}(t) + \vec{E}^{(-)}(t) = 2\,\mathrm{Re}\,\vec{E}^{(+)}(t) .$$

We decompose the amplitude of the analytic signal (1) into the horizontal and vertical components,

$$\vec{E}_0(t) = \vec{E}_H(t) + \vec{E}_V(t) , \qquad (2)$$

and introduce the parameter

$$S_0(t) = |E_H(t)|^2 + |E_V(t)|^2 ,$$

which has the meaning of the total energy and will be later defined as the zero Stokes parameter.

The Jones vector is defined as

$$\vec{e}(t) \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \equiv \begin{pmatrix} \dfrac{E_H(t)}{\sqrt{S_0(t)}} \\ \dfrac{E_V(t)}{\sqrt{S_0(t)}} \end{pmatrix} \qquad (3)$$

It describes the instantaneous polarization state of light. It is two-dimensional, complex, and normalized: $|e(t)|^2 = 1$. Note that we chose the H-V reference system (basis).

A normalized two-dimensional complex vector should normally have 2 components. But for the Jones vector, the common phase has no meaning and is usually omitted. For this reason, it can be described by two real numbers. It is convenient to introduce them as angles $\vartheta, \varphi$:

$$\alpha = \cos\frac{\vartheta}{2}, \; \beta = e^{i\varphi}\sin\frac{\vartheta}{2}. \qquad (4)$$

Later it will be clear why we divide $\vartheta$ by 2.

***2. Different polarization states.*** Consider now the Jones vectors of different polarization states.

(a) Horizontally polarized light: according to (3), $\alpha = 1, \beta = 0$: $\vec{e}_H = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(b) Vertically polarized light: similarly, $\alpha = 0, \beta = 1$: $\vec{e}_V = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(c) Diagonally polarized light: the H and V components of $\vec{E}_0(t)$ are in-phase and have equal amplitudes (Fig. 1), therefore, $\alpha = \beta = \frac{1}{\sqrt{2}}$ and $\vec{e}_D = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

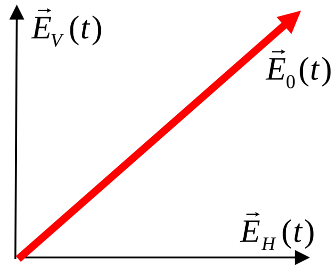

Fig.1

(d) Anti-diagonally polarized light: similarly, $\vec{e}_A = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

(e) Right circular polarized light: the V component is ahead of the H component by a phase $\frac{\pi}{2}$ (Fig.2); therefore, $\alpha = \frac{1}{\sqrt{2}}, \beta = \frac{1}{\sqrt{2}} e^{i\frac{\pi}{2}} = \frac{i}{\sqrt{2}}$ and

$\vec{e}_R = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$.

(f) Left circularly polarized light. Now, the H component is ahead, and $\vec{e}_L = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$.

Note that so far, we only consider one reference frame (one basis), HV.

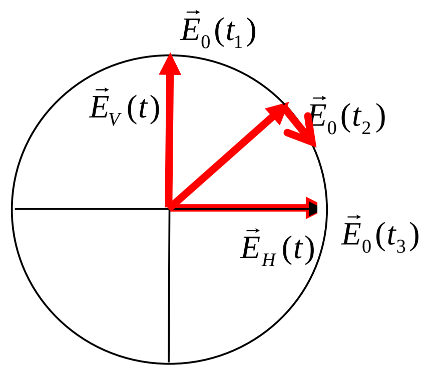

Fig.2

***3. Orthogonality of polarization states.*** At the first lecture, we discussed orthogonal polarization states. For instance, horizontally and vertically polarized light does not form interference pattern, because $\vec{e}_H \perp \vec{e}_V$. This can be easily checked by finding the inner product of vectors $\vec{e}_H = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e}_V = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. What about the right- and left- polarized states? The scalar product of their Jones vectors should be found with complex conjugation,

$$\vec{e}_R^{\,+} \cdot \vec{e}_L = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = 0 .$$

Indeed, in an interference experiment with overlapping fields $\vec{E}_R(t)$ and $\vec{E}_L(t)$, the interference term will contain $\vec{E}_R(t) \cdot [\vec{E}_L(t)]^*$.

Orthogonal are H and V polarized fields, R and L polarized fields, and any two elliptically polarized fields with opposite directions, equal aspect ratios, and orthogonal axes.

***4. Basis transformations.*** Consider now other reference frames. From linear algebra, it is known that a basis transformation of the form

$$\begin{pmatrix} \vec{e}'_1 \\ \vec{e}'_2 \end{pmatrix} = A \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}, \qquad (5)$$

for which the basic vectors stay normalized and orthogonal, $\vec{e}_i^{\,*}\vec{e}_j = \delta_{ij}$, is performed by a unitary matrix A: $A^+A=I$.

Under this transformation, the coordinates (α,β) of some Jones vector will become

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = V \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad (6)$$

where $V=(A^{-1})^T$ (transposed) or, due to the unitarity of A, $V=A^*$. Then V is also unitary.

If vectors transform according to (6), the matrices will transform as

$$D' = VDV^+. \qquad (7)$$

An example is the transformation from the HV basis to the circular one:

$$\begin{pmatrix} \vec{e}_R \\ \vec{e}_L \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \vec{e}_H \\ \vec{e}_V \end{pmatrix}. \qquad (8)$$

***5. Polarization transformations and the Jones matrices.*** We will now consider optical polarization elements. Putting aside the prisms that split light in two beams (Glan and Wollaston prisms, polarizing beamsplitters) that we will discuss later, let us now describe elements that transform the polarization state of light, maintaining the beam direction. These are: phase (ratardation) plates and polarization rotators (sugar solution, gyrotropic crystals, Faraday cells).

All these elements are lossless and they perform transformations of the Jones vector described by unitary and uni-modular 2x2 matrices (belonging to the SU(2) group), called *the Jones matrices*:

$$\vec{e}' = D\vec{e}. \qquad (9)$$

*6.* ***A phase plate*** **(*retardation plate*)** is a plate made of lossless birefringent crystal, the optic axis being in the plane of the plate. For such a plate, there are two polarization eigenstates, corresponding to the ordinary and extraordinary beams. We will later describe this situation in Lecture 5. Now, it is enough to know that the plate has 2 different refractive indices, ordinary and extraordinary, and their difference is $\Delta n \equiv n_o - n_e$. The thickness of the plate is *l*. The eigenstates of the polarization are linear, along the optic axis ζ and orthogonal to it (Fig. 3). Sometimes one uses the terms 'fast axis', 'slow axis', because the phase velocity is $c/n$.

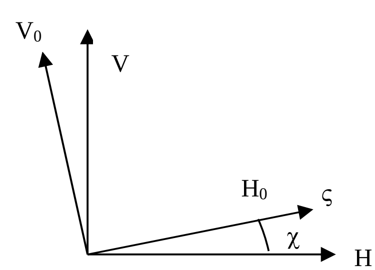

Fig. 3

Denote the basis of the plate as $\{H_0,V_0\}$, the $H_0$ axis being parallel to ζ, and let it be rotated by an angle χ with respect to the laboratory basis {H,V}.

The basis of the plate is obtained by acting on the laboratory basis with the rotation matrix *A*,

$$\begin{pmatrix} H_0 \\ V_0 \end{pmatrix} = A \begin{pmatrix} H \\ V \end{pmatrix}, \quad A = \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix}$$

To find the Jones matrix for the phase plate, we first write the Jones vector in the eigenbasis of the plate,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_0 = V \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad V = A^* = A.$$

Then, we take into account that inside the plate, the Jones vector is transformed as

$$\alpha_0 \to \alpha_0 e^{ik_e l}, \, \beta_0 \to \beta_0 e^{ik_o l}, \quad k_e = 2\pi n_e / \lambda, \quad k_o = 2\pi n_o / \lambda$$

At the output of the plate, the Jones vector in the eigenbasis has the form

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_0' = D_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_0 e^{i\pi(n_o+n_e)l/\lambda}, D_0 = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix}, \delta = \frac{\pi \Delta n l}{\lambda} = \frac{\Delta k l}{2}. \qquad (10)$$

The meaning of $\delta$ is half of the phase delay it introduces between two linear polarizations. For the future: for a $\lambda/4$ plate, $\delta=\pi/4$.

Eq. (10) simply shows that the Jones matrix is diagonal in the eigenbasis of the plate.

The phase of the Jones vector has no importance and can be dropped. Then, passing again to the laboratory basis, we get

$D=A^{-1}D_0A=A^{+}D_0A$;

$$D_0 A = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix} \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} = \begin{pmatrix} e^{i\delta}\cos\chi & e^{i\delta}\sin\chi \\ -e^{-i\delta}\sin\chi & e^{-i\delta}\cos\chi \end{pmatrix},$$

$$D = \begin{pmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} e^{i\delta}\cos\chi & e^{i\delta}\sin\chi \\ -e^{-i\delta}\sin\chi & e^{-i\delta}\cos\chi \end{pmatrix} = \begin{pmatrix} \cos\delta + i\sin\delta\cos 2\chi & i\sin\delta\sin 2\chi \\ i\sin\delta\sin 2\chi & \cos\delta - i\sin\delta\cos 2\chi \end{pmatrix},$$

or

$$D = \begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix}, \quad t = \cos\delta + i\sin\delta\cos 2\chi, \quad r = i\sin\delta\sin 2\chi \qquad (11)$$

Note that $D = \begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix}$ is the general form of a Jones matrix, for any linear lossless polarization transformation. The matrix of a beamsplitter has the same form:

$$\begin{pmatrix} E_1' \\ E_2' \end{pmatrix} = \begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}, \qquad (12)$$

where $\begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix}$ also represents the SU(2) group. The unitarity follows here from the absence of loss and the unimodularity (detD=1) is set 'by hands', from the condition that the total phase of the field (or of the Jones vector) is irrelevant.

In the case of several plates (or other elements), the matrices should be multiplied, $D = D_n \cdots D_2 D_1$.

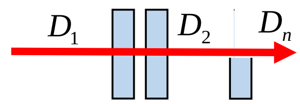


Fig.4

***7. Polarization rotator*** is a device rotating the polarization plane. This can be a Faraday cell, a sugar solution, a gyrotropic crystal, or any optically active material.

For such a material, the eigenbasis is a circular one, and the refractive indices are $n_R, n_L, \quad \Delta n \equiv n_R - n_L$. The Jones matrix in the eigenbasis is again diagonal,

$$D_0 = D_{RL} = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{pmatrix}.$$

Then in the linear basis, $D=V^{-1}D_0V=A^TD_0A^*$, where $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}$, so that

$$D = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix} \tag{13}$$

is a rotation matrix. For instance, H polarization becomes $\begin{pmatrix} \cos\delta \\ -\sin\delta \end{pmatrix}$, i.e., it is rotated by an angle δ. Clearly, (13) is also an SU(2) matrix.

An example of a rotator is crystal quartz cut orthogonal to the optic axis. Figure 5 shows the thickness needed to rotate linear polarization by $45^\circ$ (red) and $90^\circ$ (black) for different wavelengths.

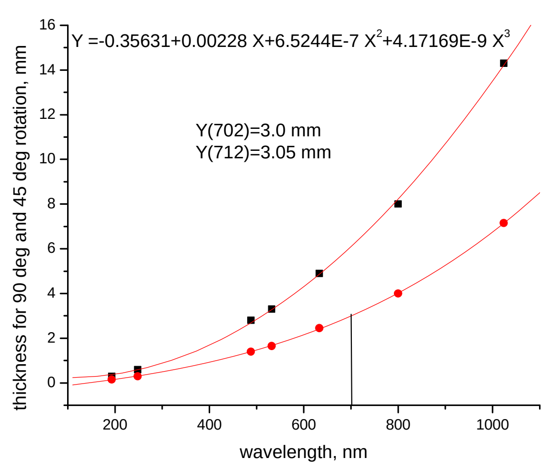


Fig. 5.

***8. Examples of polarization transformations.***

(a) Half-wave plate $\lambda/2$ (hence, $\delta=\pi/2$) at some angle $\chi$. The matrix is $D = i\begin{pmatrix} \cos(2\chi) & \sin(2\chi) \\ \sin(2\chi) & -\cos(2\chi) \end{pmatrix}$.

This is a rotation matrix.

(b) Quarter-wave plate $\lambda/4$ (hence, $\delta=\pi/4$) at $\chi=\pi/4$. The matrix is $D = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

The initial $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ becomes $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}' = D\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\sqrt{2}} + \frac{i\beta}{\sqrt{2}} \\ \frac{i\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}} \end{pmatrix}$

1. At the input, we have $\vec{e}_R = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$. Then, $|\alpha'|^2=0$. The right-circular polarization becomes V.

2. At the input, $\vec{e}_L = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$. $|\alpha'|^2=1$. The left-circular polarization becomes H.

The drawback of the Jones vector is that it is defined for the instantaneous state of polarization. It does not contain averaging and hence it cannot distinguish between polarized and unpolarized light. This is why in the next lecture we will consider the Stokes vector, which can be averaged.

Literature:


1. Born & Wolf, Principles of Optics
2. Shurcliff, Polarized light.