Lecture 3: Stokes vector and Müller matrices.

Stokes observables and Stokes parameters. Degree of polarization. The Poincare sphere. Müller matrices. Transformations with phase plates and rotators. Measurement of the Stokes observables.

As mentioned in Lecture 2, the disadvantage of the Jones vector is that it is not suitable for averaging as it is defined in terms of the field. Now, consider the Stokes vector that is free of this problem.

1. The Stokes observables (introduced in 1852). Consider first the instantaneous observables,

$$\begin{aligned} S_0 &\equiv |E_H|^2 + |E_V|^2 = I_H + I_V, \\ S_1 &\equiv |E_H|^2 - |E_V|^2 = I_H - I_V, \\ S_2 &\equiv 2\,\mathrm{Re}(E_H^* E_V), \\ S_3 &\equiv 2\,\mathrm{Im}(E_H^* E_V). \end{aligned} \tag{1}$$

In other words, instead of 2 complex numbers E_H, E_V we introduced 4 real numbers. But there is a relation:

$$|S_1|^2 + |S_2|^2 + |S_3|^2 = |S_0|^2 .$$

Therefore, one can introduce the *normalized instantaneous Stokes vector*

$$\vec{\sigma} \equiv \vec{S} / S_0, \vec{S} \equiv \{S_1, S_2, S_3\} . \tag{2}$$

Its components are related to the ones of the Jones vector as

$$\sigma_1 = |\alpha|^2 - |\beta|^2; \sigma_2 = 2\,\mathrm{Re}(\alpha^* \beta); \sigma_3 = 2\,\mathrm{Im}(\alpha^* \beta) . \tag{3}$$

We will introduce the *Stokes parameters* as the mean values

$$\begin{aligned} \langle S_0 \rangle &\equiv \langle |E_H|^2 \rangle + \langle |E_V|^2 \rangle = \langle I_H \rangle + \langle I_V \rangle, \\ \langle S_1 \rangle &\equiv \langle |E_H|^2 \rangle - \langle |E_V|^2 \rangle = \langle I_H \rangle - \langle I_V \rangle, \\ \langle S_2 \rangle &\equiv 2\,\mathrm{Re}\langle E_H^* E_V \rangle, \\ \langle S_3 \rangle &\equiv 2\,\mathrm{Im}\langle E_H^* E_V \rangle. \end{aligned} \tag{4}$$

Note that there is a more instructive form for S_2 and S_3. Because $E_{R,L} = \dfrac{E_H \pm iE_V}{\sqrt{2}}$ and $E_{D,A} = \dfrac{E_H \pm E_V}{\sqrt{2}}$, one can write $E_H^* E_V = \dfrac{1}{2}\left[|E_D|^2 - |E_A|^2\right] + \dfrac{i}{2}\left[|E_L|^2 - |E_R|^2\right]$. Therefore,

$$\begin{aligned} S_2 &= |E_D|^2 - |E_A|^2 = I_D - I_A, \\ S_3 &= |E_L|^2 - |E_R|^2 = I_L - I_L, \end{aligned} \tag{5}$$

and similarly for the mean values.

2. The degree of polarization can be now introduced as

$$P \equiv \frac{\sqrt{\langle S_1 \rangle^2 + \langle S_2 \rangle^2 + \langle S_3 \rangle^2}}{S_0} = \sqrt{\langle \sigma_1 \rangle^2 + \langle \sigma_2 \rangle^2 + \langle \sigma_3 \rangle^2} \tag{6}$$

If $P = 1$, light is polarized; if $P = 0$, light is unpolarized. The intermediate case $0 < P < 1$ corresponds to partially polarized light.

3. The Poincare sphere (1892). From (3), and from the expressions for the Jones vector components (Lecture 2, Eq. 4),

$$\alpha = \cos\frac{\theta}{2},\ \beta = e^{i\phi}\sin\frac{\theta}{2},$$

we get

$\sigma_1 = \cos\theta,\ \sigma_2 = \sin\theta\cos\phi,\ \sigma_3 = \sin\theta\sin\phi$. Here, $0 \le \theta < \pi, 0 \le \phi < 2\pi$. Thus, we introduced spherical coordinates for the Stokes vector (Fig.1).

The corresponding unity sphere is called the *Poincare sphere* (note the different orientation w.r.t. the 'standard' coordinates). Consider now the 'geography' on this sphere.

1. The equator corresponds to $\phi = 0, \phi = \pi$, which means $\sigma_3 = 0$. Then, there is no phase difference between α and β, which means linear polarization.

In particular, $\vec{\sigma} = (1,0,0)$ corresponds to H-polarization, $\vec{\sigma} = (-1,0,0)$, to V-polarization, $\vec{\sigma} = (0,1,0)$, to D-polarization, and $\vec{\sigma} = (0,-1,0)$, to A-polarization.

2. The poles. The North Pole has $\vec{\sigma} = (0,0,1)$. This means $\theta = \phi = \pi/2$ and R-polarization. The South pole has $\vec{\sigma} = (0,0,-1)$. This means $\theta = \pi/2$, $\phi = 3\pi/2$, and L-polarization.

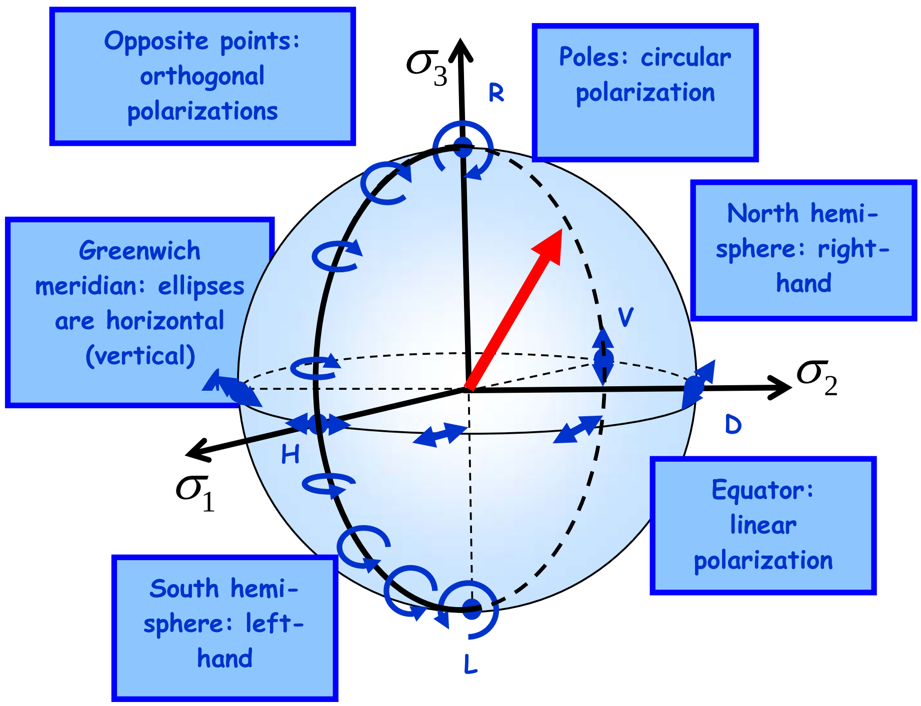

Fig.1

Generally, opposite points on the sphere correspond to orthogonal polarization states (Fig.1). The Southern hemisphere has left-hand polarization, the North one has right-hand polarization. A single meridian has the axes of ellipses parallel. On the Greenwich meridian, they are oriented along H-V.

With time, a point can 'wander' over the sphere. This describes the partially polarized light: one needs to average the Stokes vector. Alternatively, one can imagine that, instead of a single point, there is a set of points; then averaging goes over the ensemble rather than over time. If these points (or trajectories) cover part of the sphere, light is partially polarized. If they cover all the sphere, light is unpolarized.

4. Müller matrices describe how the Stokes vector is transformed under the action of polarization elements. We know, from Lecture 2, that the Jones vector components change as

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = D \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

with $D = \begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix}$.

Due to relation (3) between the Jones and Stokes vectors, one can infer that the Stokes vector transforms as

$$\vec{\sigma}' = M\vec{\sigma}, \qquad (7)$$

where M is called *the Müller matrix*. Its elements are functions of t and r, the matrix is a real (because the Stokes vector is real) 3x3 matrix, and, similar to the Jones matrix, it is unitary and unimodular. It conserves lengths of vectors and angles between two Stokes vectors. Moreover, one can show that it transforms a 'right-hand' rectangular triplet of vectors into another 'right-hand' triplet. This means that a Müller matrix describes rotations on the Poincare sphere.

To describe such a rotation, there are at least two alternative ways. One is to specify three Euler angles. The other (which we will follow) is to specify the axis of rotation and the angle of rotation. They can be found from the elements of the matrix M.

5. Transformations with phase plates. For a phase plate described in Lecture 2, t and r are given by two parameters: the phase of the plate δ and the angle of orientation χ. From the elements of the matrix, one can find that such a plate performs the rotation of the Stokes vector by an angle 2δ around an axis that lies in the equatorial plane at an angle 2χ to the σ_1 axis. This transformation is shown in Fig. 2.

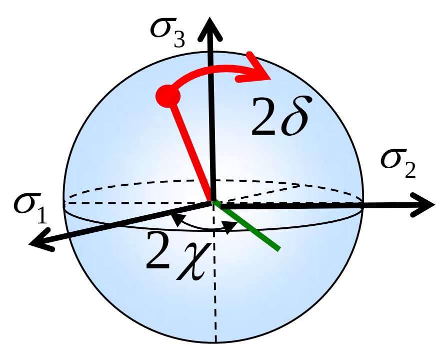

Fig.2

1. A half-wave plate (HWP) will then always rotate by an angle π. It means that it will always transform linearly polarized light (a point on the equator) into linearly polarized light (another point on the equator). The larger the χ angle, the larger the rotation on the equator (4χ). And then the polarization rotation is by angle 2χ (note that angles are always doubled on the Poincare sphere).

If the state is originally R, the plate will transform it into L, regardless of the orientation of the plate. The trajectory will be of course different – but so far it does not matter for us. It will be important later, when we discuss the geometrical phase.

2. A quarter-wave plate (QWP) will rotate the Stokes vector by an angle $\pi/2$. It is clear now how it transforms a linear polarization (H) into right-circularly polarized R: it should be oriented at $\chi = \pi/4$, then the rotation will be around the σ_2 axis. The point H in Fig.1 will then be transformed into point R. Similarly, point V will go into point L.

If the plate is oriented at a different angle (assume $\chi < \pi/4$), then point H will be transformed into a point on a lower latitude, and the resulting polarization will be elliptical.

3. A combination of several plates will result in a combination of rotations: $D = D_1 D_2 \cdots D_n$ and $M = M_1 M_2 \cdots M_n$. Any point on the Poincare sphere can be accessed from any point on the Poincare sphere by a universal system: QWP+HWP+QWP. The angles of course should be chosen according to the initial and final states. The transformation is as follows: given the initial (in the general case, elliptically polarized) state, the first QWP transforms it into some linearly polarized state; then the HWP changes this linear polarization into another linear, and the second QWP achieves the final elliptically polarized state. The same system can be realized with fiber loops.

In some special cases, of course it is enough to have two plates: HWP and QWP. For instance, if we start from a linearly polarized state, these two plates are enough.

Note that the same transformation between two points on the Poincare sphere can be realized by different rotations, corresponding to different trajectories. An example was given above, with the HWP transforming R into L through different trajectories.

The same rotation can be considered differently; one can say that a point rotates with respect to the Poincare sphere, or one can say that the sphere rotates in the opposite direction. We will use this in the description of the Stokes measurement.

6. Transformation with a polarization rotator. In Lecture 2, we obtained the Jones matrix for a polarization rotator. It had the form

$$D = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix}$$

From this matrix, one can calculate the corresponding Müller matrix and find the rotation it describes. The rotation is around the σ_3 axis, by an angle 2δ (Fig. 3). A rotator can then move any point along the equator, and it would leave unchanged the poles (R and L polarization states).

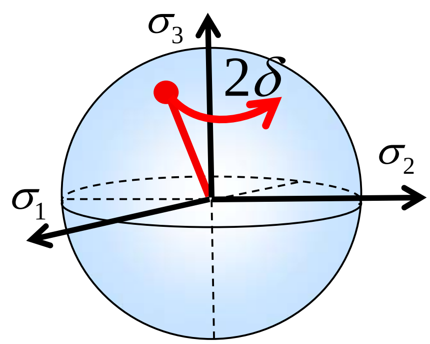

Fig.3

7. Measurement of the Stokes observables.

From the definition (4), one immediately sees that the first two Stokes observables S_0, S_1 can be measured in the setup shown in Fig. 4: a polarizing prism splits a beam in two; the horizontally and vertically polarized beams are measured by two detectors, and then, by summing and subtracting their readings, one can measure S_0, S_1, respectively, and then σ_1 as

$$\sigma_1 = \frac{i_H - i_V}{i_H + i_V}, \qquad (8)$$

where i_H, i_V are the currents (the readings) of the detectors.

From Eq. (5), one can see that the Stokes observables S_2, S_3 can be obtained in the same way, as long as we can split the beam into two beams with D and A polarizations (for S_2 measurement) and into R and L polarized beams (for S_3 measurement). But we already know how to do it. Indeed, a HWP at an angle $\chi = \pi/8$ will transform into H polarization what was previously D polarization, and simultaneously it will transform into V what was previously A. Then, a polarizing prism will split the beam into H and V; this means that the outputs will be the initial D and A.

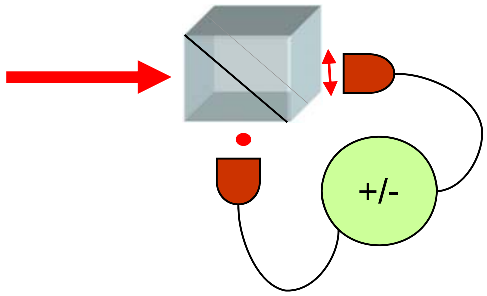

Fig. 4

The same result can be achieved with a rotator having $\delta = \pi/4$. And, of course, the simplest solution will be to rotate the prism by $\pi/4$ (Fig. 5).

For measuring S_3, we will need a QWP oriented at $\chi = \pi/4$. The beams that are polarized H and V at its output will be the initial L and R (Fig. 5).

This strategy can be considered in a different way; by placing a HWP oriented at $\chi = \pi/8$ in front of a setup as shown in Fig. 4, we rotate the Poincare sphere by an angle π around an axis that is at $2\chi = \pi/4$ to σ_1. This transformation exchanges the axes σ_1 and σ_2 and therefore transforms the measurement of S_1 into the measurement of S_2. Similarly, a QWP oriented at $\chi = \pi/4$ rotates the Poincare sphere by an angle $\pi/2$ around the σ_2 axis and therefore transforms the measurement of S_1 into the measurement of S_3.

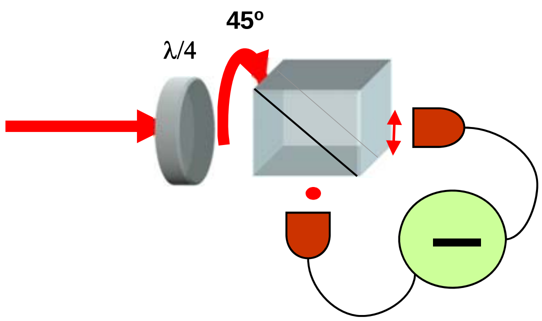

Fig. 5

Apart of measuring the mean values of S_0, S_1, S_2, S_3 and $\sigma_1, \sigma_2, \sigma_3$, we will be able to see the noise (fluctuations) of all these observables provided that our detectors are fast enough.

In principle, one can measure all Stokes observables simultaneously, by splitting the beam in parts and then using different setups for each part. But this is possible only in classical optics. In quantum optics, the Stokes observables correspond to the Stokes operators, which do not commute. As a result, their simultaneous measurement is impossible. An example is measurement of the polarization for a single photon: out of the 6 detectors necessary for performing all three Stokes measurement, only a single one will click, which is not enough.

Literature:

Shurcliff, Polarized light, Sections 2.3, 7.6.