Lecture 4: Geometric phase in polarization optics.

Geometric phase: examples with the Foucault's pendulum, with an 'optical tower' and with light propagation in a bent fiber. Decomposing a polarization state in two orthogonal and non-orthogonal states. The Pancharatnam phase.

In this lecture, we will discuss a phenomenon that appears in different fields of physics. In polarization optics, it is known as the Pancharatnam phase; it is also known as the Berry phase in quantum optics. More general terms are: topological phase, geometric phase. We will start from simple examples.

1. Simple examples.

The Foucault pendulum. Imagine a very large pendulum, called the Foucault pendulum, which is often used to demonstrate the rotation of the Earth. A lecturer usually starts it and marks the oscillation plane. After the pendulum swings for several minutes, one sees that the oscillation plane has rotated. A naïve expectation would be that in 24 hours, it will return to the initial position. But in reality, the plane of rotation will turn by an angle (Fig. 1)

$$\beta = 2\pi(1-\sin\gamma)\,. \qquad (1)$$

Here, γ is the latitude of the point where the pendulum swings. For instance, at the North Pole $\gamma = \pi/2$ and the naïve picture will be correct: the plane of the pendulum oscillations will rotate for the observer with the same angular velocity as the Earth and everything will recover in 24 hours. But at the equator, $\gamma = 0$, an observer will not see any effect of the Earth rotation, and the angle (1) will be 2π. One can notice that

$$\beta = \Omega\,, \qquad (2)$$

which is the solid angle that would be covered if the pendulum were taken along the latitude.

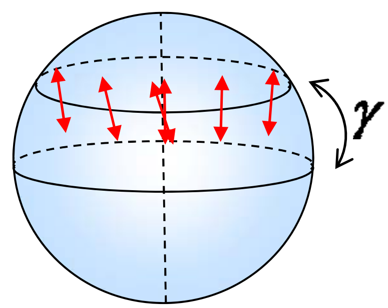

Fig.1

Another example: let the pendulum be started at the North Pole, taken to the equator along some meridian, coinciding with the direction of oscillations, then moved along the equator by an angle $\pi/2$ and returned to the North Pole (Fig. 2) – imagine that it happens very quickly, and the pendulum is still swinging. To understand what happens, we have to shift the vector describing the oscillations so that it is parallel to itself but still tangent to the surface of the Earth. And this leads to its rotation, upon return to the North Pole, by an angle $\beta = \pi/2$. Again, this is the solid angle covered by the pendulum on the surface of the Earth.

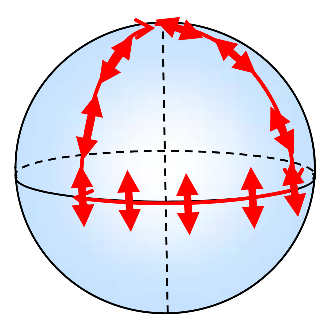

Fig.2

All this happens because, while a vector (of the pendulum oscillations), is shifted parallel to itself, it has to be tangent to a curved space. This is why the phase (angular) shift appears.

An optical tower is another simple example, already involving polarization of light. This is a device for rotating the polarization of a beam without any polarization elements, just by mirrors. The beam, initially propagating along the x axis and polarized along the z axis (vertically), is reflected by 3 mirrors. After mirror 1, it rotates by 90°

around the z axis, then mirror 2 rotates it by 90° around the x axis, then mirror 3 rotates it by 90° around the y axis. As a result, it becomes again parallel to the x axis, but now it is horizontally polarized (Fig. 3, left panel).

This $\pi/2$ rotation of polarization can be also related to some solid angle covered in a 3D space. Indeed, the right panel of Fig. 3 shows the evolution of the k vector of the beam. Initially along the x axis, it rotates by a solid angle $\pi/2$ and returns again to the same direction. As a result, the angle of polarization rotation is again given by the solid angle covered in 3D space (on a sphere), see (2).

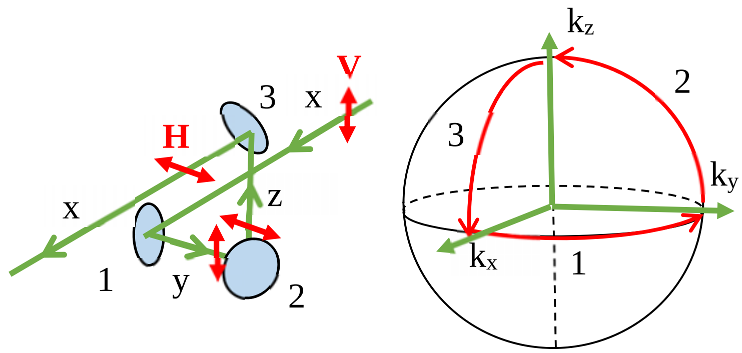

Fig.3

This can be a problem in experiments where polarization should be maintained: any reflections of the beam that are not within a single plane will spoil the polarization state of light.

Propagation in a fiber. This situation can be generalized to the case of light propagating in an optical fiber. If the fiber lies in one plane, the polarization will not be changed. But if the fiber direction forms a trajectory in the 3D space, then the polarization will rotate by an angle given by the solid angle covered by this trajectory.

2. Interference of arbitrary polarized beams.
We mentioned, in the first lectures, that beams in orthogonal polarization states do not form an interference pattern. But what about non-orthogonally polarized states? Consider points A,B on the Poincare sphere (Fig. 4): if they are not oppositely placed on the Poincare sphere, they describe polarization states that are not orthogonal. Then, if we try to observe interference between such fields, the visibility will be nonzero. What will it be?

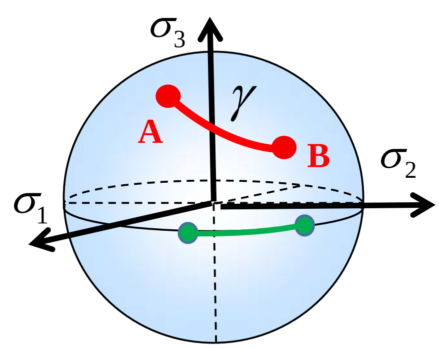

Fig. 4

Let the Jones vectors of the two beams be $\vec{e}_A, \vec{e}_B$; then the total intensity will be given by

$$I \propto \left|\vec{e}_A + \vec{e}_B\right|^2 = 2(1 + \left|\vec{e}_A^*\vec{e}_B\right| \cos\delta), \qquad (3)$$

where $\delta = \arg(\vec{e}_A^*\vec{e}_B)$ is the phase of the interference (as defined by Pancharatnam). Clearly, the visibility, defined as

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}},$$

is given by $\left|\vec{e}_A^*\vec{e}_B\right|$. In other words, the interference visibility for two polarized beams of equal intensity is given by the absolute value of the scalar product of their Jones vectors.

This scalar product absolute value has a clear interpretation on the Poincare sphere. Indeed, if A and B correspond to linearly polarized states (green points in Fig. 4), with coordinates $\theta_{A,B}$ (for linearly polarized states, $\phi_{A,B} = 0$), then the scalar product is

$$\vec{e}_A^* \vec{e}_B = \cos(\frac{\theta_A}{2})\cos(\frac{\theta_B}{2}) + \sin(\frac{\theta_A}{2})\sin(\frac{\theta_B}{2}) = \cos\frac{\theta_A - \theta_B}{2}. \qquad (4)$$

But $\theta_A - \theta_B = \gamma$, the length of the arc between the two green points in Fig. 4. (Note that two points on a sphere can be connected by many arcs; this one is the shortest, *the geodesic line*.) But Eq. (3) can be generalized: for any two points A,B with the angular separation γ (red points), the visibility of interference will be given by the cosine of half the angular distance between them:

$$V = \cos\frac{\gamma}{2}. \qquad (5)$$

For instance, the visibility will be zero for opposite points and $1/\sqrt{2}$ for points separated by a quadrant.

3. Decomposing a beam in two orthogonally polarized components.

Thus, the length of the arc between any two points A, B on the Poincare sphere (*the geodesic line*) determines the modulus of the Jones vectors scalar products: $\left|\vec{e}_A^* \vec{e}_B\right| = \cos\frac{\gamma}{2}$. Any beam given by a point C on the Poincare sphere, with the Jones vector $\vec{e}_C$, can be decomposed into orthogonally polarized beams, given by points A and A' (Fig. 5). The projections will be $\left|\vec{e}_A^* \vec{e}_C\right| = \cos\frac{\alpha}{2}$ and $\left|\vec{e}_{A'}^* \vec{e}_C\right| = \cos\frac{(\pi - \alpha)}{2} = \sin\frac{\alpha}{2}$.

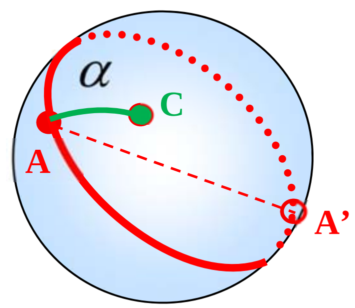

Fig. 5

4. Decomposing a beam in two non-orthogonal components. Similarly, a beam in polarization state C can be decomposed in two beams in polarization states A and B, the angular distance between them being γ (Fig. 6). The relation between the Jones vectors will be:

$$\vec{e}_C \sin\frac{\gamma}{2} = \vec{e}_A \sin\frac{\beta}{2} + \vec{e}_B \sin\frac{\alpha}{2}. \qquad (6)$$

This is similar to decomposing a usual vector in a 2D Cartesian space into two non-orthogonal components. Indeed, in Fig. 7, if **A**, **B**, **C** are unit vectors, then for the projection a of **C** on **A** we have $a\sin\gamma = \sin\beta$, and for the projection b of **C** on **B** we have $b\sin\gamma = \sin\alpha$, a similar relation. The only difference is that on the Poincare sphere all angles should be divided by 2.

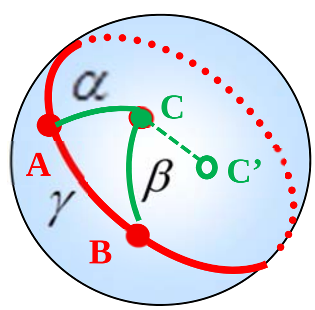

Fig. 6

5. Pancharatnam phase.

Taking squared modulus of (6), we get

$$\sin^2\frac{\gamma}{2} = \sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} + 2\left|\vec{e}_A^* \vec{e}_B\right| \cos\delta \sin\frac{\alpha}{2}\sin\frac{\beta}{2} = \sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} + 2\cos\frac{\gamma}{2}\cos\delta\sin\frac{\alpha}{2}\sin\frac{\beta}{2}.$$

Then, the Pancharatnam phase is

$$\cos\delta = \frac{\sin^2\frac{\gamma}{2} - \sin^2\frac{\alpha}{2} - \sin^2\frac{\beta}{2}}{2\cos\frac{\gamma}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}.$$

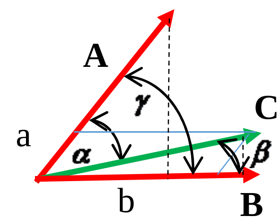

Fig. 7

Then,

$$-\cos\delta = \frac{\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} - \sin^2\frac{\gamma}{2}}{2\cos\frac{\gamma}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}$$

If the point C' is considered instead, opposite to C, for which $\alpha' = \pi - \alpha$ and $\beta' = \pi - \beta$, then

$$\cos(\pi - \delta) = \frac{\cos^2\frac{\alpha'}{2} + \cos^2\frac{\beta'}{2} + \cos^2\frac{\gamma}{2} - 1}{2\cos\frac{\gamma}{2}\cos\frac{\alpha'}{2}\cos\frac{\beta'}{2}}. \qquad (7)$$

But this, according to equations of solid geometry, is cosine of half the solid angle formed by points A, B, C' on the sphere:

$$\cos(\pi - \delta) = \cos\frac{\Omega'}{2} \quad . \qquad (8)$$

But $\Omega' = 2\pi - \Omega$, where Ω is the solid angle formed by points A,B,C on the sphere. It follows that

$$\delta = \frac{\Omega}{2}. \qquad (9)$$

It means that *if two beams in polarization states A and B form interference pattern in the polarization state C, the phase of the interference will be given by half of the solid angle covered by the geodesic triangle ABC.*

In particular, if the interference of states A and B leads to a state C lying on the geodesic connecting them, the phase of interference is zero.

This means that if we move a state along the geodesic line on the Poincare sphere, no phase shift appears until we make a half-circle. But if we move it along a closed contour ABC in Fig. 6, then there appears a phase shift, the Pancharatnam phase (9).

6. Experiment on observing the Pancharatnam phase.

A simplest experiment is to make the point go a full circle around the equator, by rotating polarization with the help of a rotator, by 2π. A solid angle $\Omega = 2\pi$ This means that, say, linear V polarization becomes linear H, then returns to linear V polarization, but the field vector changes the direction to opposite (Fig. 8). Because we ignore the total phase of the Jones vector, we say that the polarization state did not change. But then this is a phase that has appeared, and if we make this beam interfere with the initial beam, a phase shift of $\beta = \pi$ will appear, which is exactly $\Omega/2$.

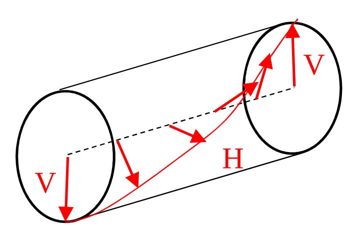

Fig. 8

A more interesting experiment is to put half-wave plates into both arms of a Mach-Zehnder interferometer and send there a circularly polarized light. For instance, it is R-polarized. The plates will transform this state into L-polarization, and at the output there will be perfect interference pattern (with 100%) visibility. But the phase of the interference will depend on the orientation of the plates. If one of them is oriented with the optic axis horizontal, the rotation of the other one will change the solid angle as shown in Fig. 8.

7. Other manifestations of the geometric phase.

There are numerous manifestations, all having the following features:

1. The phase does not depend on the trajectory, but only on the surface surrounded by it.
2. The appearance of this phase is due to the constraint imposed on a vector moving along a trajectory on a curved surface.

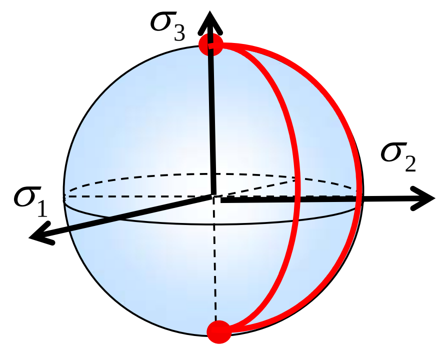

Fig. 8

Manifestations of the geometric phase include:

1. A spin ½ particle in magnetic field, whose representation on the Bloch sphere is completely equivalent to the representation of the polarized light on the Poincare sphere. The phase acquired by it is a particular case of the Berry's phase.
2. The Aharonov-Bohm effect: a current can take one of the two trajectories going from point A to point B; if there is magnetic field between these two trajectories, then there is interference with the phase determined by the magnetic field flux through the contour formed by the trajectories. This principle is used in quantum magnetometers where two superconducting junctions are connected parallel, and the magnetic flux between them is measured very accurately.

Literature:

1. S. Pancharatnam, Generalized Theory of Interference and its Applications. Proc. Indian Acad. Sci. A 44, 247 (1956).
2. D N Klyshko, Berry geometric phase in oscillatory processes. Phys.-Usp. **36**, 1005 (1993).