Lecture 6: Polarization optics of crystals - 2.

The normal surface and the ray surface. Isotropic, uniaxial and biaxial crystals. Spatial and temporal walk-off. A liquid crystal.

1. The normal surface and the ray surface.

The task formulated in the end of the last lecture was: for a given direction of the wavevector find the two possible values of the phase velocity. This surface is given by the equation (17) of the previous lecture,

$$\frac{k_x^2}{\dfrac{1}{\varepsilon_x}-\dfrac{1}{n^2}}+\frac{k_y^2}{\dfrac{1}{\varepsilon_y}-\dfrac{1}{n^2}}+\frac{k_z^2}{\dfrac{1}{\varepsilon_z}-\dfrac{1}{n^2}}=0. \tag{1}$$

After substituting $v=\dfrac{c}{n}$ and denoting $v_i^2 \equiv \dfrac{c^2}{\varepsilon_i}, i=x,y,z$, we get

$$\frac{k_x^2}{v^2-v_x^2}+\frac{k_y^2}{v^2-v_y^2}+\frac{k_z^2}{v^2-v_z^2}=0. \tag{2}$$

This dependence determines the so-called *normal surface*: for each direction of the wavevector two values of v (we showed in the previous lecture that there are two solutions) are plotted. This surface is more complicated than an ellipse and it consists of two shells.

Similarly, one can derive the equation for the ray surface. It takes the form

$$\frac{S_x^2}{\dfrac{1}{u^2}-\dfrac{1}{u_x^2}}+\frac{S_y^2}{\dfrac{1}{u^2}-\dfrac{1}{u_y^2}}+\frac{S_z^2}{\dfrac{1}{u^2}-\dfrac{1}{u_z^2}}=0, \tag{3}$$

and it gives the two possible values of the group velocity for a given direction of the Poynting vector.

2. Optical types of crystals.

Depending on the symmetry, the crystals can belong to 3 classes: isotropic, uniaxial, and biaxial.

Isotropic crystals are most symmetric; they can only have the cubic structure. For such crystals, $\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon_0$, Eq. (17) has only one solution, and there is no birefringence.

Uniaxial crystals are less symmetric; here belong crystals with trigonal, tetragonal, and hexagonal lattice. They have $\varepsilon_x = \varepsilon_y = \varepsilon_0 \neq \varepsilon_z$, two different values of the dielectric constant. Equation (1) becomes

$$\left(\frac{1}{\varepsilon_0}-\frac{1}{n^2}\right)\left[k_x^2+k_y^2\left(\frac{1}{\varepsilon_z}-\frac{1}{n^2}\right)+k_z^2\left(\frac{1}{\varepsilon_0}-\frac{1}{n^2}\right)\right]=0. \tag{4}$$

The z direction will be further called the optic axis. We will introduce the spherical coordinates for the wavevector:

$$k_x = k\sin\vartheta\cos\varphi,\, k_y = k\sin\vartheta\sin\varphi,\, k_z = k\cos\vartheta .$$

Then, (4) has two solutions:

$$\frac{1}{\varepsilon_0} - \frac{1}{n^2} = 0;$$

$$\sin^2 \vartheta \left(\frac{1}{\varepsilon_z} - \frac{1}{n^2} \right) + \cos^2 \vartheta \left(\frac{1}{\varepsilon_0} - \frac{1}{n^2} \right) = 0.$$

For the refractive index, this means

$$n = \sqrt{\varepsilon_0};$$

$$n = \frac{1}{\sqrt{\dfrac{\sin^2 \vartheta}{\varepsilon_z} + \dfrac{\cos^2 \vartheta}{\varepsilon_0}}}. \tag{5}$$
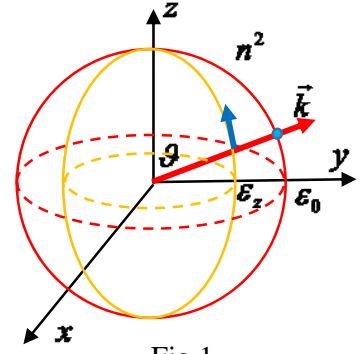

Fig.1

These two solutions give two different surfaces. The first one is a sphere (red in Fig. 1); for any direction of $\vec{k}$ there is the same solution for the refractive index. This is the ordinary wave. The second one is an ellipsoid (orange in Fig. 1), which has a circle in the xy-cut and an ellipse in the z-cut (because the refractive index depends only on the angle between $\vec{k}$ and z). This is the extraordinary wave. For a wavevector along the z axis, there is just a single value of the refractive index, $n_0 = \sqrt{\varepsilon_0}$. The optic axis is therefore the direction along which two Fresnel surfaces intersect.

Sometimes, $n_0 = \sqrt{\varepsilon_0}$ is called the ordinary index, n_o. The other index is called extraordinary one, $n_e = \sqrt{\varepsilon_z}$. A more familiar form for the refractive index of the extraordinary wave is

$$\frac{1}{n^2} = \frac{\sin^2 \vartheta}{n_e^2} + \frac{\cos^2 \vartheta}{n_o^2}. \tag{6}$$

Figure 1 shows a situation where $n_o > n_e$. This type of uniaxial crystal is called negative; the opposite case, $n_o < n_e$, is called a positive crystal.

Biaxial crystals is the most general case; this class includes triclinic, monoclinic, and orthorhombic crystals. For them, $\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z$, and the Fresnel equation has the most general form (1). In this case, one can only say that there are two different shells of the Fresnel surface, which intersect in 4 points (2 for each optic axis).

However, if we consider one of the principal planes, i.e., one of the planes where $k_i = 0$, Eq. (1) simplifies. For instance, for the plane orthogonal to k_z, we have either

$$\frac{1}{\varepsilon_z} - \frac{1}{n^2} = 0, \tag{7}$$

or

$$\frac{k_x^2}{\dfrac{1}{\varepsilon_x} - \dfrac{1}{n^2}} + \frac{k_y^2}{\dfrac{1}{\varepsilon_y} - \dfrac{1}{n^2}} = 0.$$

Then, multiplying by the product of the denominators, we get the equation of an ellipse

$$k_x^2 \left(\frac{1}{\varepsilon_y} - \frac{1}{n^2} \right) + k_y^2 \left(\frac{1}{\varepsilon_x} - \frac{1}{n^2} \right) = 0. \tag{8}$$

Both solutions (7) and (8) are shown in Fig.2 by red and yellow lines, respectively. Here we assumed that $n_y < n_z < n_x$. For this reason, the two curves (the ellipse and the circle) intersect. The four points where they intersect define the two optic axes.
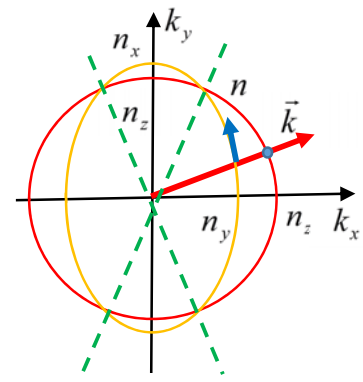
The same situation occurs in the two other principal planes; however, due to the condition $n_y < n_z < n_x$ there will be no more intersections of the ellipse and the circle. In the plane orthogonal to k_y (shown in Fig. 3), the ellipse is outside the circle, while in the plane orthogonal to k_x, the circle is outside.
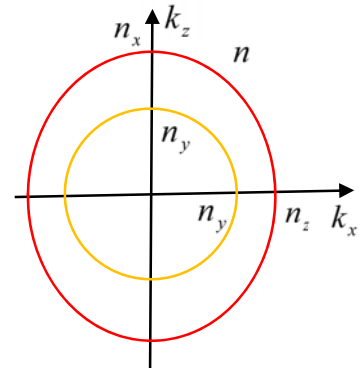

Fig.2

We can combine this information about the three principal planes, each one containing an ellipse and a circle, and draw a 3D plot of two shells intersecting only at 4 points determining 2 optic axes.

3. Direction of the Poynting vector.

One can prove (Ref. 1) the following important statement:
For each direction of the wavevector, the direction of the electric field is tangent to the corresponding Fresnel surface. Consequently, the Poynting vector is orthogonal to it. The electric field vectors are shown in Fig.1 and Fig.2 by blue arrows for extraordinary waves and by blue dots for ordinary waves. We see that for the ordinary wave in a uniaxial crystal as well as for the circle in a biaxial crystal (in a principal plane, it is also the ordinary wave), the Poynting vector and the wavevector are parallel. However, for the shell that has an elliptic cross-section with the principal plane (extraordinary wave), it is not the case.

Fig.3

4. Spatial and temporal walk-off.

The Poynting vector shows the direction of the energy transfer; it is also called the ray vector. Let us now explain the situation we discussed at the very first lecture: if an unpolarized or arbitrary polarized beam enters a calcite crystal (Fig. 4) at some angle (neither 0 nor 90 degrees) with the optic axis, shown by a black arrow, it splits in two beams which are at an angle inside the crystal and then are parallel outside it, but shifted.
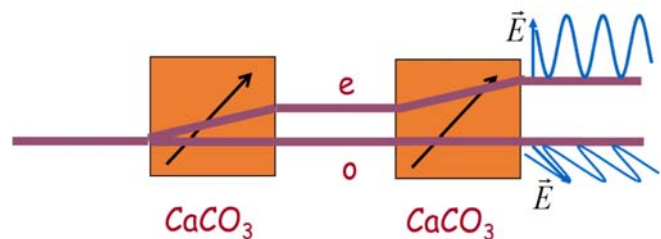

Fig.4

The explanation is clear from Fig. 1. Calcite is a negative uniaxial crystal, with $n_o = 1.66$ and $n_e = 1.49$ at 532 nm. This large difference (about 15%) between the refractive indices leads to a very stretched ellipse (6) for the extraordinary wave.

Using the above rule (that the electric field is along the tangent to the ellipse), one can find the anisotropy angle α by differentiating (6) in ϑ:

$$n \tan\alpha = \frac{dn}{d\vartheta}.$$

(9)

We get

$$\frac{-2}{n^3}\frac{dn}{d\vartheta}=\frac{2\sin\vartheta\cos\vartheta}{n_e^2}-\frac{2\sin\vartheta\cos\vartheta}{n_o^2}.$$

As a result,

$$\tan\alpha=\frac{1}{2}\sin(2\vartheta)n^2\left(\frac{1}{n_o^2}-\frac{1}{n_e^2}\right).\tag{10}$$

One can simplify it to the form

$$\tan(\alpha\pm\vartheta)=\pm\frac{n_o^2}{n_e^2}\tan\vartheta,\tag{11}$$

where the upper signs are for negative crystals and lower signs, for positive crystals (derive this result at the problem class).

For the ordinary wave, $\alpha=0$ and the Poynting vector is along the wavevector. This is why the splitting of e and o beams appears. It is most pronounced, as it is clear from (10) and (11), for $\vartheta=45°$. This effect is called *the spatial walk-off*, and the anisotropy angle is called the walk-off angle.

Similarly to how the spatial walk-off is due to the non-collinearity of the phase and group velocity vectors, *temporal walk-off* appears because their absolute values differ. In this lecture, we neglect dispersion, and therefore in the absence of spatial walk-off there will be no temporal walk-off as well (u and v differ by just $\cos\alpha$).

5. Circular birefringence. Gyrotropy.

So far, we discussed the directions of the E vector for various crystals, given by the tangent to the Fresnel surface. We saw that two directions exist for every wavevector. This means two linear polarization states. What about circular polarization?

It turns out that even in usual crystals, in the direction of the optic axis there is still not one value of the refractive index, n_0, but two again, corresponding to complex values of the electric field and therefore to right and left circular polarizations. The two values differ by a tiny amount $\Delta n_c=n_R-n_L$. Such crystals are called gyrotropic, and the effect, gyrotropy. An example of a gyrotropic crystal is quartz.

Circular birefringence exists also in liquids – an example is sugar solution. In this case, polarization rotation is performed by each molecule; such molecules are called chiral. For each molecule, the polarizability differs depending on whether light is right- or left-circularly polarized. Therefore, the dielectric constant depends on the sign of the circular polarization as well (through the Lorentz-Lorenz law):

$$\frac{\varepsilon-1}{\varepsilon+2}=\frac{1}{3}N\alpha,$$

where N is the density of molecules.

6. Liquid crystals.

A liquid crystal consists of molecules for which polarizability behaves the same way as the dielectric function in the case of a usual crystal: its value depends on the direction of the wavevector with respect to the direction of the molecule. (The molecules in a liquid crystal are stretched.) In addition, all molecules have the same or about the same orientation, and this orientation can be changed by applying the electric field. Then, with a certain direction of light

hitting a cuvette with a liquid crystal, one can change the refractive index by applying electric field.

These properties of liquid crystals are very important for making spatial light modulators, which will be considered in the next lecture.

Literature:
1. M. Born and E. Wolf, Principles of Optics, sections 14.1-14.3.
2. R. W. Boyd, Nonlinear optics, section 2.1.