Lecture 9: Polarization in quantum optics.

Polarization modes: photon creation and annihilation operators. Stokes operators. Single-photon state. Quantum measurement of the Stokes observables.

Now we will re-consider the description of polarization we used before. In quantum optics, every physical value corresponds to some operator. So now we will have not Stokes observables but Stokes operators. Not electric fields but creation and annihilation operators. And these operators will act on states – but this will be the subject of the next lecture. Another notion, common between classical and quantum optics, will be polarization modes. I assume that you have studied quantum mechanics, but not necessarily quantum optics.

1. Polarization modes. Creation and annihilation operators.

We have to recall a bit of quantum optics. Usually in quantum optics one considers a single mode of light. How do these modes appear? One imagines a `quantization box', just a very large box such that nothing interesting happens outside, all fields outside are zero. Then one can imagine that all fields are periodic in x,y,z, with the period given by the box size – and then we are left with a discrete set of wavevectors

$$\vec{k}_{lmn} = \{k_x; k_y; k_z\} = \frac{2\pi}{L}\{l, m, n\}. \qquad (1)$$

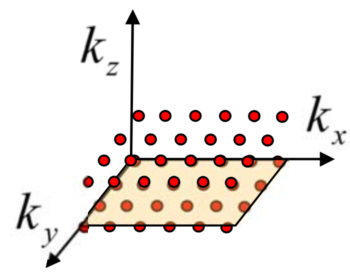


Fig. 1

Usually, only a single mode, given by l,m,n, is considered. But, according to the previous lectures, for every such a mode, i.e., for every direction of the $\vec{k}$ vector, there are two polarization states. Then for completeness, one should write

$$\vec{k} = \{k_x; k_y; k_z; \sigma\},$$

where σ denotes the polarization. Depending on the situation, they can be linear, or circular, or elliptic, but should be orthogonal. One chooses them for convenience: in a crystal, one should use o and e. But in the lab, it is convenient to use H and V. So we will now consider one wavevector mode and two polarization modes: $\sigma = H, V$. Or, $\sigma = D, A$; or $\sigma = L, R$.

The complex fields in classical optics, negative-frequency one $E^{(-)}$ and positive-frequency one, $[E^{(-)}]^* = E^{(+)}$, in quantum optics become operators, $E^{(-)} \to \hat{E}^{(-)}$ and $E^{(+)} \to \hat{E}^{(+)}$. These operators are, in their turn, written as a superposition over modes.

$$\hat{\vec{E}}^{(+)}(z) = \sum_{\vec{k}} \hat{\vec{E}}_{\vec{k}}^{(+)} e^{i\vec{k}\vec{r} - i\omega t} = \sum_{\vec{k}} c_{\vec{k}} a_{\vec{k}} e^{i\vec{k}\vec{r} - i\omega t}, \qquad (2)$$

where $a_{\vec{k}}$ is the photon annihilation operator. But here we consider only a single wavevector mode; then, there are only 2 mode operators, $\hat{\vec{E}}_H^{(+)}, \hat{\vec{E}}_V^{(+)}$ and their Hermitian conjugates $\hat{\vec{E}}_H^{(-)}, \hat{\vec{E}}_V^{(-)}$. They are proportional to the dimensionless photon creation operators a_H^+, a_V^+.

These operators are not Hermitian: $a_H^+ \neq a_H$. They also do not commute:

$$a_H a_H^+ - a_H^+ a_H = 1, a_V a_V^+ - a_V^+ a_V = 1 \; . \qquad (3)$$

At the same time, the operators for different modes do commute:

$$a_H a_V^+ = a_V^+ a_H, \; a_V a_H = a_H a_V \; . \qquad (4)$$

This is because the modes are independent.

Photon creation and annihilation operators form the photon-number operator:

$$\hat{N}_H = a_H^+ a_H, \hat{N}_V = a_V^+ a_V \quad . \tag{5}$$

The photon-number operator is Hermitian,

$$\hat{N}_H^+ = \hat{N}_H, \hat{N}_V^+ = \hat{N}_V .$$

In classical optics, it corresponds to intensity:

$$I_H = E_H^{(-)} E_H^{(+)}, I_V = E_V^{(-)} E_V^{(+)} \quad . \tag{6}$$

2. Stokes operators

In quantum optics, instead of the Stokes parameters we introduce the Stokes operators by changing intensities to photon-number operators:

$$\begin{aligned} &\hat{S}_0 \equiv \hat{N}_H + \hat{N}_V, \quad \hat{S}_1 \equiv \hat{N}_H - \hat{N}_V, \\ &\hat{S}_2 \equiv \hat{N}_D - \hat{N}_A = a_D^+ a_D - a_A^+ a_A = a_H^+ a_V + a_V^+ a_H, \\ &\hat{S}_3 \equiv \hat{N}_L - \hat{N}_R = a_L^+ a_L - a_R^+ a_R = (a_H^+ a_V - a_V^+ a_H)/i. \end{aligned} \tag{7}$$

The Stokes operators are Hermitian, by definition, and they correspond to real observables.

Note that the choice of the mode (H,V,D,A,L,R) is arbitrary; they are all equivalent but orthogonal pairwise. What does orthogonality of two polarization modes mean?

(1) There is no interference between fields in these modes;
(2) Fields in these modes are statistically independent;
(3) (Quantum) The corresponding photon creation and annihilation operators commute: $[a_H^+, a_V] = [a_L^+, a_R] = ... = 0$, the corresponding observables can be measured simultaneously and so on.

The term ‘Stokes parameters’ will further refer to the *mean values* of the Stokes operators.

Commutation and uncertainty relations. In quantum optics, operators corresponding to different observables sometimes do not commute:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \neq 0. \tag{8}$$

One can see that the Stokes operators $\hat{S}_1, \hat{S}_2, \hat{S}_3$ do not commute. To show it, we will use the following rules for commutators:

$$\begin{aligned} &[\hat{A}, \hat{B} + \hat{C}] = \hat{A}\hat{B} + \hat{A}\hat{C}, \\ &[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]. \end{aligned} \tag{9}$$

Then,

$$\begin{aligned} &[\hat{S}_1, \hat{S}_2] = [a_H^+ a_H - a_V^+ a_V, a_H^+ a_V + a_V^+ a_H] = [a_H^+ a_H, a_H^+ a_V] + [a_H^+ a_H, a_V^+ a_H] - \\ &- [a_V^+ a_V, a_H^+ a_V] - [a_V^+ a_V, a_V^+ a_H] = a_H^+ [a_H, a_H^+] a_V + a_V^+ [a_H^+, a_H] a_H - \\ &- a_H^+ [a_V^+, a_V] a_V - a_V^+ [a_V, a_V^+] a_H = a_H^+ a_V - a_V^+ a_H + a_H^+ a_V - a_V^+ a_H = 2i\hat{S}_3. \end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned} &[\hat{S}_2, \hat{S}_3] = 2i\hat{S}_1, \\ &[\hat{S}_3, \hat{S}_1] = 2i\hat{S}_2. \end{aligned} \tag{10'}$$

Meanwhile, $\hat{S}_0$ commutes with all other three operators:

$$[\hat{S}_0,\hat{S}_2] = [a_H^+ a_H + a_V^+ a_V, a_H^+ a_V + a_V^+ a_H] = [a_H^+ a_H, a_H^+ a_V] + [a_H^+ a_H, a_V^+ a_H] +$$
$$+ [a_V^+ a_V, a_H^+ a_V] + [a_V^+ a_V, a_V^+ a_H] = a_H^+ [a_H, a_H^+] a_V + a_V^+ [a_H^+, a_H] a_H + \quad (11)$$
$$+ a_H^+ [a_V^+, a_V] a_V + a_V^+ [a_V, a_V^+] a_H = a_H^+ a_V - a_V^+ a_H - a_H^+ a_V + a_V^+ a_H = 0.$$

Similarly,

$$[\hat{S}_0,\hat{S}_1] = [\hat{S}_0,\hat{S}_3] = 0. \quad (11')$$

From these, the uncertainty relations follow. Indeed, one can show that the uncertainties of non-commuting operators (8), $\Delta A \equiv \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$ and $\Delta B \equiv \sqrt{\langle \hat{B}^2 \rangle - \langle \hat{B} \rangle^2}$, satisfy the condition

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [\hat{A},\hat{B}] \rangle \right|. \quad (12)$$

Accordingly, the Stokes operators satisfy the uncertainty relation

$$\Delta S_i \Delta S_j \geq \left| \langle S_k \rangle \right|, \quad k \neq i \neq j. \quad (13)$$

Physically, it means that different Stokes observables cannot be measured simultaneously unless one of the three is zero.

3. A single-photon state

In quantum optics, to measure the mean value of some observable A means to find the average of the corresponding operator $\hat{A}$ over a state. If the averaging is over a pure state $|\Psi\rangle$, it is

$$\langle \hat{A} \rangle = \langle \Psi | \hat{A} | \Psi \rangle. \quad (14)$$

The averaging over a mixed state $\hat{\rho}$ is written as

$$\langle \hat{A} \rangle = Sp(\hat{A}\hat{\rho}), \quad (15)$$

but we will only deal with a pure state, namely with a single-photon state. It is a special case of a Fock state $|N\rangle$, the eigenstate of the photon-number operator:

$$\hat{N}|N\rangle = N|N\rangle. \quad (16)$$

The single-photon state is the one with the eigenvalue 1,

$$\hat{N}|1\rangle = 1 \cdot |1\rangle. \quad (17)$$

But we will consider two different single-photon states, $|1\rangle_H$ and $|1\rangle_V$, eigenstates of $\hat{N}_H$ and $\hat{N}_V$, and their superposition of the general form,

$$|\Psi\rangle = \alpha|1\rangle_H + \beta|1\rangle_V, \quad |\alpha|^2 + |\beta|^2 = 1. \quad (18)$$

The vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ looks like the Jones vector we considered earlier, but it has a different meaning now. But as before, $\alpha = 1$ corresponds to an H-polarized state, $\alpha = 0$ to a V-polarized state, the common phase does not matter etc.

4. Measurement of the Stokes observables

The 'canonical' definition of measurement in quantum mechanics is performing a projection on the eigenstates of the corresponding operator: for instance, $\hat{A}|A\rangle = A|A\rangle$. A Hermitian operator always has a complete orthonormal set of eigenstates: $\hat{A}|A_n\rangle = A_n|A_n\rangle, \langle A_m|A_n\rangle = \delta_{mn}$. Any state can be decomposed over this set,

$$|\Phi\rangle = \sum_n c_n|A_n\rangle, \tag{19}$$

and to measure the mean value of $\hat{A}$ over $|\Phi\rangle$ one should project $|\Phi\rangle$ onto the set $\{|A_n\rangle\}$. Then,

$$\langle\hat{A}\rangle = \langle\Phi|\hat{A}|\Phi\rangle = \sum_m c_m^* \sum_n c_n A_n \langle A_m|A_n\rangle = \sum_n |c_n|^2 A_n \ . \tag{20}$$

Every term in this sum is the probability that the state is $|A_n\rangle$, $P_n \equiv |c_n|^2| = \langle\Phi|A_n\rangle|^2$, times the value of the operator $\hat{A}$ in this state.

The same can be done using the so-called expansion of the unity. For any complete set $\{|A_n\rangle\}$, one can write

$$\hat{1} = \sum_n |A_n\rangle\langle A_n|. \tag{21}$$

Then,

$$\langle\hat{A}\rangle = \langle\Phi|\sum_n \hat{A}|A_n\rangle\langle A_n||\Phi\rangle = \sum_n A_n\langle\Phi||A_n\rangle\langle A_n||\Phi\rangle = \sum_n A_n\,|\langle\Phi||A_n\rangle|^2 = \sum_n A_n P_n.$$

Let us do this procedure for the Stokes operators. First, what are the eigenstates of the Stokes operators? Let us find them for $\hat{S}_1$. We have to solve the equation

$$\hat{S}_1|\Psi\rangle = s|\Psi\rangle.$$

Substituting (18), we get

$$[\hat{N}_H - \hat{N}_V][\alpha|1\rangle_H + \beta|1\rangle_V] = \alpha s|1\rangle_H + \beta s|1\rangle_V .$$

Using (17), we obtain

$$[\alpha|1\rangle_H - \beta|1\rangle_V] = \alpha s|1\rangle_H + \beta s|1\rangle_V ,$$

or

$$\alpha(1-s)|1\rangle_H = \beta(1+s)|1\rangle_V .$$

Because the states are orthogonal, the factors in front should be zero,

$$\alpha(1-s) = \beta(1+s) = 0 .$$

We get two possibilities:

(1) $s = 1, \beta = 0 \Rightarrow |1\rangle_H$

(2) $s = -1, \alpha = 0 \Rightarrow |1\rangle_V$.

We obtained that the eigenstates of $\hat{S}_1$ are horizontally polarized single photon and vertically polarized single photon. Similarly (do it at the problem class), the eigenstates of $\hat{S}_2$ are $|1\rangle_D, |1\rangle_A$, also with the eigenvalues 1, -1. The eigenstates of $\hat{S}_3$ are $|1\rangle_L, |1\rangle_R$, also with eigenvalues 1, -1. For measuring a certain Stokes observable, one should perform the projection on the corresponding eigenstate. This is done with the setup shown in Fig.2; for instance, for   measurement there should be just the polarizing prism and two 'click' detectors. If the upper detector clicks, we say that the photon is V polarized, and write down the result $S_1 = -1$ (Fig.2). Then the

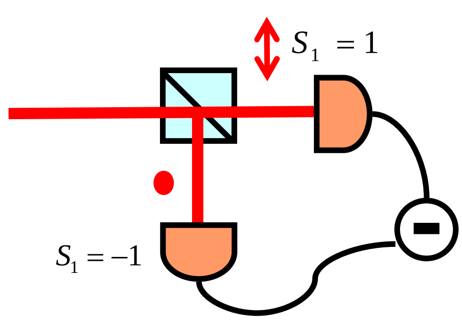


Fig.2

mean value (the first Stokes parameter) is calculated by averaging the results of all tries: $\langle S_1 \rangle = \frac{1}{n}\sum_{i=1}^{n} S_{1i}$. The same way the variance can be calculated.

Measurement of the other Stokes parameters will be similar (with a waveplate in front of the prism).

This is a projection measurement: we expand a unity over the eigenstates of the Stokes observable,

$$\hat{1} = |1\rangle_H \langle 1|_H + |1\rangle_V \langle 1|_V ,$$

$$\langle \hat{S}_1 \rangle = 1 \cdot |\langle \Psi | 1 \rangle_H|^2 - 1 \cdot |\langle \Psi | 1 \rangle_V|^2 \equiv P_H - P_V .$$

Clearly, we will get for any Stokes observable a value $|\langle \hat{S}_i \rangle| \le 1$.

Note that if the photon is polarized neither horizontally nor vertically (for instance, $|1\rangle_D$), it will be reflected or transmitted with some probability (50% for $|1\rangle_D$), and its detection does not tell us much. The fact that the photon was detected, say, in the 'transmitted path' tells us only that it was not polarized vertically.

This clearly demonstrates the impossibility to measure simultaneously non-commuting Stokes observables. In classical optics, a beam can be split in 3 and in each one, a different Stokes parameter is measured. But for a quantum state, such is a single-photon one, this is principally impossible as the photon will be absorbed in just one setup.

The scheme from Fig. 2 can be also realized with a piece of calcite crystal (or any other crystal with large birefringence) cut at ~45° to the optic axis (Fig. 3). Then the Stokes observables can be measured through walk-off: if a photon is shifted, it is V-polarized; if it is not shifted, it is H-polarized.

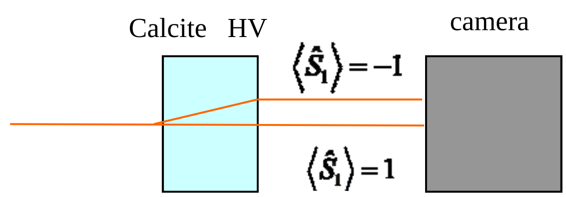


Fig. 3

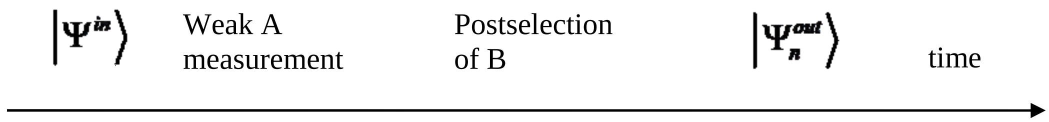
4. Weak measurement of the Stokes observables

This idea appeared rather long ago [Aharonov, Albert, Vaidman, PRL 60, 1351 (1988)], but experiments are carried out only recently. According to this idea, even non-commuting observables can be measured simultaneously if the measurement device interacts with them weakly. This weak measurement can be best demonstrated with the Stokes observables: in Fig. 3, the beam should be broader than the displacement due to the walk-off. The result of this weak measurement can have the absolute value larger than any of the Stokes eigenvalues.

The scheme is as follows. First, observable A is measured weakly. Then, the state is projected on the eigenstates of another operator (B), which does not commute with A.

Let the initial state be $|\Psi^{in}\rangle$. The weak measurement of A yields $\langle \hat{A} \rangle = \langle \Psi^{in} | \hat{A} | \Psi^{in} \rangle$, but we plug here the expansion of the unity in the eigenstates of B: $\hat{1} = \sum_n |\Psi_n^{out}\rangle\langle\Psi_n^{out}|$, $\hat{B}|\Psi_n^{out}\rangle = b_n|\Psi_n^{out}\rangle$.

And then we postselect one of these eigenstates:



$$\langle\hat{A}\rangle = \langle\Psi^{in}|\sum_n |\Psi_n^{out}\rangle\langle\Psi_n^{out}|\hat{A}|\Psi^{in}\rangle = \sum_n \langle\Psi^{in}||\Psi_n^{out}\rangle\langle\Psi_n^{out}|\hat{A}|\Psi^{in}\rangle = \sum_n \left|\langle\Psi_n^{out}||\Psi^{in}\rangle\right|^2 \frac{\langle\Psi_n^{out}|\hat{A}|\Psi^{in}\rangle}{\langle\Psi_n^{out}||\Psi^{in}\rangle}.$$

This can be interpreted as the expansion over 'weak values of A',

$$\langle\hat{A}\rangle = \sum_n P_n A_{weak}^n,$$

$$P_n = \left|\langle\Psi_n^{out}||\Psi^{in}\rangle\right|^2, \quad A_{weak}^n \equiv \frac{\langle\Psi_n^{out}|\hat{A}|\Psi^{in}\rangle}{\langle\Psi_n^{out}||\Psi^{in}\rangle}.$$

Finally, we project on a single n, for instance, $n=0$. Then,

$$\langle\hat{A}\rangle = A_{weak}^0 \equiv \frac{\langle\Psi_0^{out}|\hat{A}|\Psi^{in}\rangle}{\langle\Psi_0^{out}||\Psi^{in}\rangle}.$$

Due to the denominator, which can be very small if the output and input states are almost orthogonal, this weak value can be very large. In particular, it can be larger than any of $\hat{A}$ eigenvalues.

As an example, let us first prepare an eigenstate of $\hat{S}_1$, say, $|H\rangle$, then weakly measure the operator $(\hat{S}_1+\hat{S}_2)/\sqrt{2}$, and then project the state onto an eigenstate of $\hat{S}_2$. For instance, we postselect $|D\rangle$. Then,

$$\langle(\hat{S}_1+\hat{S}_2)/\sqrt{2}\rangle = A_{weak}^0 \equiv \frac{\langle D|(\hat{S}_1+\hat{S}_2)/\sqrt{2}|H\rangle}{\langle D||H\rangle} = \frac{\langle D|(|H\rangle+|V\rangle)}{\sqrt{2}\cdot 1/\sqrt{2}} = \sqrt{2} > 1.$$

Such an experiment is shown in Fig. 4. The displacement of the beam in the direction 22.5° will correspond to $\frac{(\hat{S}_1+\hat{S}_2)}{\sqrt{2}}$. It will be twice larger than the displacement of a thin beam and larger by a factor $\sqrt{2}$ than the eigenvalue of any Stokes eigenvalue.

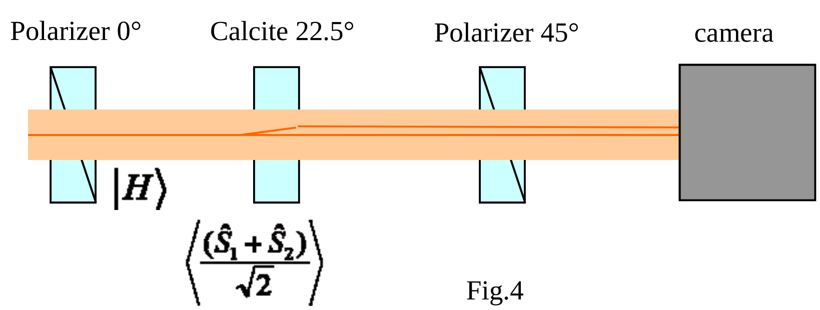


Fig.4

Literature:

1. B. A. Robson, The Theory of Polarization Phenomena.
2. Jeff Z. Salvail, Megan Agnew, Allan S. Johnson, Eliot Bolduc, Jonathan Leach, and Robert W. Boyd, Full characterization of polarization states of light via direct measurement. Nature Photonics, 2013