Lecture 7. Discrete-variable and continuous-variable approaches.

Quantum measurement. Coincidence counting, HBT and HOM experiments, quasi-probabilities, homodyne detection.

1. Quantum measurement.

How to describe the measurement? How can we find the mean value of an observable, for instance, the photon number, and what will be the measurement accuracy?

This measurement can be performed by a photon-counting detector, which provides the mean number of photons $\langle N\rangle = \langle\Psi|\hat{N}|\Psi\rangle$. Depending on the state, the result will be different. Similarly, one can calculate the variance and the standard deviation. This calculation predicts the result of measuring a quantity. Fig.2 shows, for instance, the result of measuring the number of photons. It fluctuates from try to try, or with time.

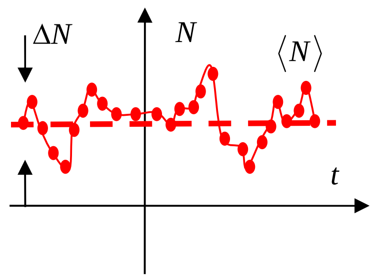


Fig.2

For instance, we have found at the last lecture that the mean number of photons in a coherent state is $\langle N\rangle = |\alpha|^2$, and the variance is $\Delta N^2 = |\alpha|^2 = \langle N\rangle$ (shot noise).

Projection postulate and projective (von Neumann) measurement. What if a system is in an eigenstate of a certain operator and we need to calculate the observable corresponding to another operator? For instance, again, the system is in state $|\alpha\rangle$ and we need to calculate the probability to have a certain number of photons N.

Because (the expansion of a unity) $\sum_N |N\rangle\langle N| = \mathbf{1}$, one can write

$$\hat{N} = \sum_N \hat{N}|N\rangle\langle N|, \qquad (1)$$

and

$$\langle N\rangle = \langle\alpha|\hat{N}|\alpha\rangle = \sum_N \langle\alpha|\hat{N}|N\rangle\langle N|\alpha\rangle = \sum_N N\langle\alpha|N\rangle\langle N|\alpha\rangle \equiv \sum_N P(N|\alpha)N,$$

where $P(N|\alpha) = |\langle N|\alpha\rangle|^2$ gives the probability to have N photons in a coherent state. This probability can be calculated (see the previous lecture) as

$$|\langle N|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2N}}{N!}. \qquad (2)$$

It is given by the Poissonian distribution. (Again, a Poissonian distribution has the variance equal to the mean, this is why we got the shot noise.)

Generally, the probability $P(A|B)$ to measure an eigenvalue A of an operator $\hat{A}$ in an eigenstate $|B\rangle$ of an operator $\hat{B}$ is $|\langle A\|B\rangle|^2$.

2. Discrete variables and continuous variables.

In quantum optics community, strange enough, there are two manners of research, which overlap not very much. The first one deals with the measurements and variables associated with photon numbers. The second one deals with quadratures, everything is calculated in terms of quadratures, and quadrature-related values are measured. These two approaches gradually merge, as there appear more and more groups working at the boundary. Still, I would like to focus on these two traditional approaches.

3. Discrete-variable approach: operators and correlation functions

In the discrete-variable approach, the main operator is the photon-number one,

$$\hat{N} \equiv a^+ a.$$

Furthermore, its powers are of interest, and especially important are normally-ordered powers,

$$:\hat{N}^k :\equiv (a^+)^k a^k.$$

The values to measure are the mean values of these operators, i.e., Glauber's correlation functions (CFs) of different orders,

$$G^{(k)} \equiv \langle : N^k : \rangle,$$

including the CFs with different time and space arguments,

$$G^{(k)}(t_1,...t_k, r_1,...r_k) \equiv \langle : \hat{N}(t_1, r_1)...\hat{N}(t_k, r_k) : \rangle,$$

and including normalized CFs:

$$g^{(k)}(t_1,...t_k, r_1,...r_k) \equiv \frac{g^{(k)}(t_1,...t_k, r_1,...r_k)}{\langle N(t_1, r_1)\rangle ... \langle N(t_k, r_k)\rangle}.$$

There are two very important properties of these CFs:

(1) Normalized CFs are not sensitive to losses.

Let us first prove this statement. The losses, as is usual in quantum optics, we will describe as a beamsplitter (BS) with the amplitude transmission and reflection coefficients $t, r, \quad |t|^2 + |r|^2 = 1$. One of the inputs contains only vacuum; in the Heisenberg approach we will describe what happens using the operator transformation, which has the form

$$\begin{pmatrix} a' \\ v' \end{pmatrix} = \begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix} \begin{pmatrix} a \\ v \end{pmatrix}, \qquad (4)$$

where we denoted the operator in the ‚vacuum' mode as v (Fig.1).

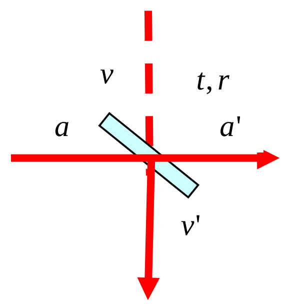


Fig.1

Then the output photon-number operator has the form

$$\hat{N}' = |t|^2 \hat{N} + |r|^2 \hat{N}_v + (t^* r a^+ v + h.c.). \qquad (5)$$

As we proved in Lecture 6,

$$:\hat{N}^k := \hat{N}(\hat{N}-1)....(\hat{N}-k+1).$$

After the BS,

$$:\hat{N}'^k := \hat{N}'(\hat{N}'-1)....(\hat{N}'-k+1).$$

Now, let us average this expression over the state $|\Psi\rangle_a |0\rangle_v$. Due to the averaging over $|0\rangle_v$ all terms in (5) except the first one vanish because they contain vacuum-state operators. Then,

$$\langle : (\hat{N}')^k : \rangle = |t|^{2k} \langle : \hat{N}^k : \rangle.$$

And of course a similar equality is valid for the mean photon number:

$$\left\langle \hat{N}' \right\rangle = |t|^2 \left\langle \hat{N} \right\rangle.$$

Then, for the normalized k[th]-order CF, we have

$$g'^{(k)} = \frac{\left\langle :(\hat{N}')^k: \right\rangle}{\left\langle :\hat{N}': \right\rangle^k} = g^{(k)},$$

and we see that losses do not change the normalized CF.

(2) Normalized CFs are sensitive to the number of modes.

As an example consider the second-order CF for light containing K independent modes with the same statistics (normalized CF g) and the same mean photon number N_0. Then, the total photon-number operator is

$$\hat{N} = \sum_{i=1}^{K} \hat{N}_i \ ,$$

the mean photon number is KN_0, and the normalized second-order CF is

$$g^{(2)} = \frac{\left\langle :[\sum_{i=1}^{K} \hat{N}_i]^2: \right\rangle}{(KN_0)^2} = \frac{K\left\langle :\hat{N}_i^{\,2}: \right\rangle + K(K-1)\left\langle \hat{N}_i \ \hat{N}_{j\neq i} \right\rangle}{(KN_0)^2} = \frac{KgN_0^{\,2} + K(K-1)N_0^{\,2}}{(KN_0)^2} = 1 + \frac{g-1}{K}.$$

The same result will be obtained for an effective number K of modes.

4. Correlation functions for different states

Coherent state. Obviously, for a coherent state $|\alpha\rangle$,

$$\left\langle :\hat{N}^k: \right\rangle = |\alpha|^{2k} \text{ and } g^{(k)} = 1.$$

For *a Fock state* $|N\rangle$,

$$\left\langle :\hat{N}^k: \right\rangle = N(N-1)\cdots(N-K+1) \text{ and } g^{(k)} = 1(1-\frac{1}{N})\cdots(1-\frac{K-1}{N}).$$

In particular, $g^{(2)} = 1 - \frac{1}{N}$.

Let us now consider a *thermal state*, with the mean photon number n,

$$\rho_{th} = \sum_{N=0}^{\infty} p_N |N\rangle\langle N|, \quad p_N = \frac{1}{1+n}\kappa^N, \quad \kappa \equiv \frac{n}{n+1}.$$

The mean photon number will be obtained by averaging a Fock-state mean photon number with the classical probability,

$$\langle N \rangle = \sum_{N=0}^{\infty} p_N N = \sum_{N=0}^{\infty} \frac{1}{1+n}\kappa^N N = \frac{1}{1+n}\sum_{N=0}^{\infty}\kappa\frac{\partial}{\partial\kappa}\kappa^N = \frac{1}{1+n}\kappa\frac{\partial}{\partial\kappa}\sum_{N=0}^{\infty}\kappa^N = \frac{1}{1+n}\kappa\frac{\partial}{\partial\kappa}\frac{1}{1-\kappa} = \frac{\kappa}{1+n}\frac{1}{(1-\kappa)^2} = n.$$

Similarly,

$$\left\langle :N^2: \right\rangle = \sum_{N=0}^{\infty} p_N N(N-1) = \sum_{N=0}^{\infty}\frac{1}{1+n}\kappa^N N(N-1) = \frac{1}{1+n}\sum_{N=0}^{\infty}\kappa^2\frac{\partial^2}{\partial\kappa^2}\kappa^N = \frac{2\kappa^2}{1+n}\frac{1}{(1-\kappa)^3}.$$

Then,

$$g^{(2)} = \frac{2\kappa^2}{(1+n)}\frac{(1+n)^2(1-\kappa)^4}{(1-\kappa)^3\kappa^2} = 2(1+n)(1-\kappa) = 2(1+n)(1-\frac{1}{1+n}) = 2.$$

We got the same result as in the classical approach.

5. Discrete-variable approach: measurement.

Photon counting is the main experimental technique in discrete-variable quantum optics. We already mentioned it in Lecture 3; the main characteristic of a single-photon detector is its QE η, and the mean number of counts m is given by $\eta\langle\hat{N}\rangle$. Here, I will always consider a single-mode detector, i.e., such that the detection volume is equal to the coherence volume. The non-unity quantum efficiency affects photodetection the same way as losses; in fact, the way it is described in theory is by assuming that an ideal ($\eta = 1$) detector is preceded by a BS with $t = \sqrt{\eta}, r = \sqrt{1-\eta}$. Then, the factorial moments of photocounts number will have a simple relation to the factorial moments of the photon number:

$$G_m^{(k)} \equiv \langle m(m-1)...(m-k+1)\rangle = \eta^k \langle :\hat{N}^k:\rangle \equiv \eta^k G^{(k)}.$$

In particular,

$$G_m^{(2)} \equiv \langle m(m-1)\rangle = \langle m^2\rangle - \langle m\rangle = \Delta m^2 - \langle m\rangle + \langle m\rangle^2,$$

hence the normalized CF can be measured as

$$g^{(2)} = g_m^{(2)} = 1 + \frac{\Delta m^2 - \langle m\rangle}{\langle m\rangle^2}.$$

So the bunching parameter can be calculated from the measured variance and mean of the photocount number.
As we remember, the classical requirement is $g^{(2)} \geq 1$.
Anti-bunching, thus, is a nonclassical feature.

Coincidence counting. However, in experiment it is much simpler to use another method of CF measurement – namely, the Hanbury Brown-Twiss technique (Lecture 3). Let us recall the figure (Fig.2), where this time we consider the 'vacuum' input. The probability that detector 1 fires at time t_1 and detector 2 fires at time t_2 is given by the average of the operator

$$\hat{E}_1^{(-)}(t_1)\hat{E}_2^{(-)}(t_2)\hat{E}_1^{(+)}(t_1)\hat{E}_2^{(+)}(t_2).$$

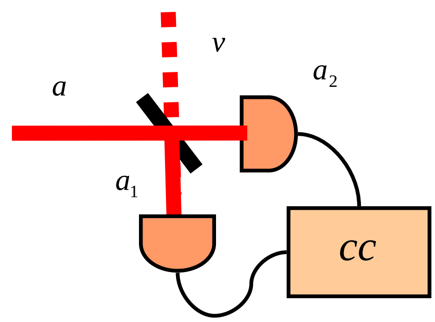


Fig.2

As we learned at Lecture 5, negative-frequency field operators can be written in terms of photon creation operators. Strictly speaking, frequency integration is involved here, but for simplicity we will assume the time moments equal and also simply replace the field operators with the photon creation and annihilation operators. Then, the coincidence counting rate will be proportional to

$$R_c \sim \langle a_1^+ a_2^+ a_1 a_2\rangle T_c,$$

where T_c is the coincidence window (the larger this window, the more probable is a coincidence), while the mean count rates in both detectors will scale as $R_1 \sim \langle a_1^+ a_1\rangle$ and $R_2 \sim \langle a_2^+ a_2\rangle$. Then,

$$\frac{R_c}{R_1 R_2 T_c} = \frac{\langle a_1^+ a_2^+ a_1 a_2\rangle}{\langle a_1^+ a_1\rangle\langle a_2^+ a_2\rangle} \qquad (6)$$

(it can be proven that the proportionality coefficients are such that the equality holds true.)

On the other hand, the combination on the right-hand side of (6), with an account for the BS transformation (4), can be written as

$$\frac{\langle a_1^+ a_2^+ a_1 a_2 \rangle}{\langle a_1^+ a_1 \rangle \langle a_2^+ a_2 \rangle} = \frac{\langle (t^* a^+ + r^* v^+)(-ra^+ + tv^+)(ta + rv)(-r^* a + t^* v) \rangle}{\langle (t^* a^+ + r^* v^+)(ta + rv) \rangle \langle (-ra^+ + tv^+)(-r^* a + t^* v) \rangle}.$$

While averaging, we have to take into account that we average over a state $|\Psi\rangle_a |0\rangle_v$, and whenever a v^+ operator acts on the vector ${}_v\langle 0|$, there is a zero, and the same if v acts on $|0\rangle_v$. Then,

$$\frac{\langle a_1^+ a_2^+ a_1 a_2 \rangle}{\langle a_1^+ a_1 \rangle \langle a_2^+ a_2 \rangle} = \frac{\langle t^* a^+ (-ra^+)(ta)(-r^* a) \rangle}{\langle (t^* a^+)(ta) \rangle \langle (-ra^+)(-r^* a) \rangle} = \frac{\langle (a^+)^2 a^2 \rangle}{\langle a^+ a \rangle^2} \equiv g^{(2)}.$$

We see that the second-order normalized CF can be obtained from the measurement of coincidence counting rate and photon counting rates in two detectors. Consider now some interesting effects measured this way.

Anti-bunching. We saw that classical statistical optics 'forbids' the values of $g^{(2)} < 1$. However, this is always the case for a single-photon state, even in superposition with the vacuum: for $|\Psi\rangle = c_0|0\rangle + c_1|1\rangle$, $\langle (a^+)^2 a^2 \rangle = 0$, hence $g^{(2)} = 0$. This is also true for a mixture, $\rho = p_0 |0\rangle\langle 0| + p_1 |1\rangle\langle 1|$. And we also saw that Fock states all manifest anti-bunching.

High $g^{(2)}$. You will learn from Lecture 10 that through parametric down-conversion, one can generate two-photon light in the state $|\Psi\rangle = |0\rangle + c|2\rangle$ (here, of course, $|c| << 1$ - otherwise normalization would be needed). Then, $\langle (a^+)^2 a^2 \rangle = |c|^2$, $\langle a^+ a \rangle = |c|^2$, hence $g^{(2)} = \frac{1}{|c|^2} >> 1$.

This is called superbunching, or two-photon correlations.

Hong-Ou-Mandel effect. It is interesting to see what happens when two single-photon states arrive simultaneously at the beamsplitter. We will use again the combination

$$\frac{R_c}{R_1 R_2 T_c} = \frac{\langle a_1^+ a_2^+ a_1 a_2 \rangle}{\langle a_1^+ a_1 \rangle \langle a_2^+ a_2 \rangle},$$

only this time we will assume that there is no vacuum at the second input (Fig.3) but the input state is

$$|\Psi\rangle = |1\rangle_a |1\rangle_b = a^+ b^+ |0\rangle_a |0\rangle_b,$$

then the output state can be obtained by transforming the operators according to (4). Indeed, let us first write the inverse transformation,

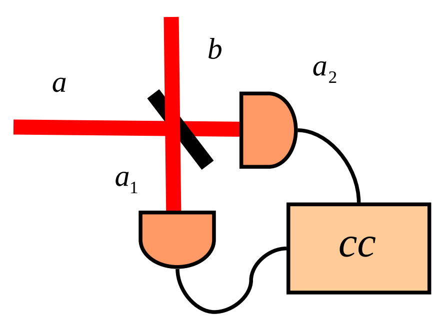


Fig.3

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix}^{-1} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix}^{+} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} t^* & -r \\ r^* & t \end{pmatrix}^{+} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Then

$$|\Psi\rangle = (ta_1^+ - r^* a_2^+)(ra_1^+ + t^* a_2^+)|0\rangle_a |0\rangle_b = [tr(a_1^+)^2 - t^* r^* (a_2^+)^2 + (|t|^2 - |r|^2) a_1^+ a_2^+]|0\rangle_a |0\rangle_b.$$

If the BS is 50%, $|t|^2 = |r|^2$, and $|\Psi\rangle = [tr(a_1^+)^2 - t^* r^* (a_2^+)^2]|0\rangle_a |0\rangle_b$.

It is clear that no coincidence counts will be observed for detectors 1 and 2. Photon pairs will go as a whole to channel 1 or to channel 2.

6. Continuous-variable approach: quadratures, quasi-probabilities.

In continuous-variable approach, in the focus are the quadrature operators,

$$\hat{x}_1 = \frac{a^+ + a}{2}, \quad \hat{x}_2 = \frac{a - a^+}{2i}.$$

The quantum state is described by their mean values and variances, but a more complete picture could be made by considering their joint probability distribution

$$P(\hat{x}_1, \hat{x}_2) - ?$$

We know that these operators do not commute; therefore such a joint distribution, strictly speaking, cannot be introduced. But there are many ways to introduce quasi-probabilities. As always in the probability theory, a probability density can be defined as the Fourier-transform of a certain characteristic function. In quantum optics, the characteristic function will be the mean value of some operator. The three quasi-probabilities we will consider will correspond simply to different ordering of a^+, a in this operator.

Glauber-Sudarshan function. A very convenient instrument in quantum optics is considering coherent states $|z\rangle \equiv |z' + iz''\rangle$ as a basis. The Glauber-Sudarshan function is introduced as the density operator in the coherent-state representation:

$$\hat{\rho} = \iint dz' \, dz'' P(z) |z\rangle\langle z|.$$

It is very convenient that in the P-representation the density matrix is diagonal. This representation has many nice properties – for instance, it provides a simple way for averaging normally-ordered operators. Indeed, consider an operator like $\hat{A} = (a^+)^n a^m$, then

$$\langle \hat{A} \rangle = Tr(\hat{A}\hat{\rho}) = \iint dz' \, dz'' P(z) \langle z | (a^+)^n a^m | z \rangle = \iint dz' \, dz'' P(z) (z^*)^n z^m. \quad (7)$$

It means that the mean of a normally-ordered operator can be calculated in the P-representation just by putting z^* everywhere for a^+ and z for a. In this representation, it is convenient to consider the characteristic function

$$C^{(n)}(w', w'') \equiv \langle : \exp\{wa^+ - w^* a\} : \rangle \equiv \langle \exp\{wa^+\} \exp\{-w^* a\} \rangle.$$

Using (7), we obtain this mean value:

$$C^{(n)}(w', w'') = \iint dz' \, dz'' P(z) \exp\{wz^*\} \exp\{-w^* z\} = \iint dz' \, dz'' P(z) \exp\{wz^* - w^* z\}.$$

This is actually 2D Fourier transformation, and the inverse is

$$P(z', z'') = \frac{1}{\pi^2} \iint dw' \, dw'' C^{(n)}(w', w'') \exp\{-wz^* + w^* z\}.$$

This P-function cannot be really considered a probability. For a coherent state $|\alpha\rangle$, it is a delta-function:

$$P(z', z'') = \delta^{(2)}(z - \alpha).$$

And for a Fock state, it is even more singular. For this reason, it cannot be measured directly.

Wigner function. It is obtained as the Fourier-transform of a symmetric characteristic function:

$$C^{(s)}(w', w'') \equiv \langle \exp\{wa^+ - w^* a\} \rangle,$$

$$W(z', z'') = \frac{1}{\pi^2} \iint dw' \, dw'' C^{(s)}(w', w'') \exp\{-wz^* + w^* z\}.$$

This is the Wigner function, usually written in terms of quadratures: $W(x_1, x_2)$. It has a remarkable property: its marginal distribution gives the correct probability density for a quadrature:

$$\int dx_2 W(x_1, x_2) = p(x_1)\ .$$

For this reason, it can be measured; we will soon see how. The Wigner function can be never singular but it can be negative, which is considered as a sign of nonclassical light (impossible for a classical probability density).

Husimi function. This is the Fourier transform of the anti-normally ordered characteristic function,

$$C^{(a)}(w', w'') \equiv \left\langle \exp\{w^* a\}\exp\{-wa^+\}\right\rangle,$$

$$Q(z', z'') = \frac{1}{\pi^2}\iint dw' dw'' C^{(a)}(w', w'')\exp\{-wz^* + w^* z\}.$$

Husimi function is used more seldom. It is never singular or negative.

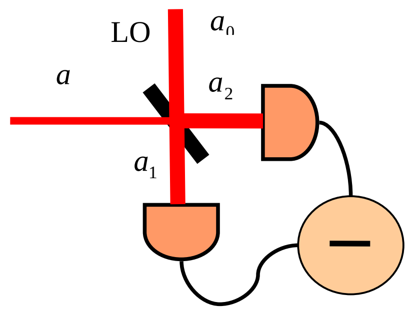


Fig.4

7. Continuous-variable approach: measurement.

Homodyne detection. The main technique for measuring quadratures is homodyne detection. Again we have a setup with the beamsplitter and two detectors, but this time no coincidence circuit (we will not multiply but subtract the signals from the detectors). This time the detectors are not counting ones, but analog ones, having high quantum efficiency and producing photocurrents: an electron is released for nearly every photon. The state under study is fed into port a, and into the other port strong coherent radiation is fed (the local oscillator, LO). Then, according to transformation (4),

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix}\begin{pmatrix} a \\ a_0 \end{pmatrix}$$

Assume that the photocurrent in each detector reflects just the flux of photons (each photon creates an electron). Also, let the BS be 50%, so $t = r = 1/\sqrt{2}$. Then, photocurrent in detector 1 is

$$\hat{i}_1 = a_1^+ a_1 = \frac{1}{2}(a^+ + a_0^+)(a + a_0) = \frac{1}{2}\{a^+ a + a_0^+ a_0 + (a_0^+ a + h.c.)\},$$

and in detector 2,

$$\hat{i}_2 = a_2^+ a_2 = \frac{1}{2}(-a^+ + a_0^+)(-a + a_0) = \frac{1}{2}\{a^+ a + a_0^+ a_0 - (tr^* a_0^+ a + h.c.)\}.$$

The difference of these two photocurrents will be

$$\hat{i}_- \equiv \hat{i}_1 - \hat{i}_2 = a_0^+ a + h.c.$$

Assume now that the LO is so strong that its state can be considered classical. Then, the operator is replaced by a complex number $\alpha_0 = |\alpha_0| e^{i\phi}$, which is the amplitude of the LO, and

$$\hat{i}_- = |\alpha_0|(a^+ e^{i\phi} + h.c.),$$

Or, expressing the exponential through sines and cosines,

$$\hat{i}_- = |\alpha_0|[a^+\cos\phi + ia^+\sin\phi + a\cos\phi - ia\sin\phi] = 2|\alpha_0|[\hat{x}_1\cos\phi + \hat{x}_2\sin\phi].$$

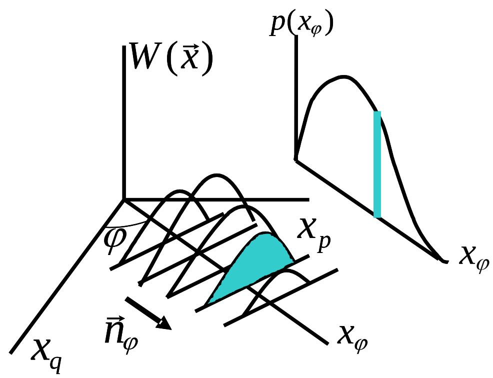
The expression in brackets represents the ‚generalized quadrature’,

$$\hat{x}(\phi) = [\hat{x}_1\cos\phi + \hat{x}_2\sin\phi]. \qquad (8)$$

Depending on the phase of the LO, it can be the first or the second quadrature.

Variance measurement. Homodyne detection is often used to observe squeezing. Then, the main observable is the variance of the quadrature, Δx^2, which is compared to the coherent-light variance. If squeezing has to be measured, it is very important that the detection has little losses.

Quantum tomography. Eq. (5) shows that depending on the LO phase, any quadrature can be measured. It turns out that by doing it for a series of phases, one can reconstruct the Wigner function by means of the Radon transformation.



Home task:
Using the transformation of operators on a beamsplitter, prove that a coherent state after a beamsplitter will remain a coherent state.

Books:

1. Klyshko, Physical foundations of quantum electronics, Sec. 7.
2. Mandel, Wolf, Optical coherence and quantum optics, Secs. 14.7, 21.6..