Lecture 4. From the Hamiltonian to a quantum state, at low and high efficiency of interaction.

Schrödinger and Heisenberg pictures in quantum optics. Weak interaction: photon pairs. Strong interaction: squeezing.

1. *Schrödinger and Heisenberg pictures.*

Now, we have to describe 'what is going on in the crystal or fiber' as a result of the Hamiltonian we derived,

$$\hat{H} = i\hbar\Gamma a_1^+ a_2^+ + h.c. \qquad (1)$$

There are two ways to describe the evolution: in terms of state vectors or in terms of operators. We either assume that the state is evolving, $|\Psi\rangle = |\Psi(t)\rangle$ and the operators are constant, $\hat{A} = \hat{A}_0$ (Schrödinger picture), or assume that the operators are evolving, $\hat{A} = \hat{A}(t)$ and the state vector is constant, $|\Psi\rangle = |\Psi_0\rangle$ (Heisenberg picture). Note that what we are really interested in are neither the states nor the operators, but measurable quantities, the result of averaging operators over states,

$$\langle A \rangle \equiv \langle \Psi | \hat{A} | \Psi \rangle \ . \qquad (2)$$

Or, for the density matrix, $\langle A \rangle \equiv Sp(\hat{\rho}\hat{A})$.

This value,

$$\langle \Psi(t) | \hat{A}_0 | \Psi(t) \rangle = \langle \Psi_0 | \hat{A}(t) | \Psi_0 \rangle, \quad Sp(\hat{\rho}(t)\hat{A}_0) = Sp(\hat{\rho}_0\hat{A}(t)),$$

will be invariant to the choice of the picture, as we will see further.

In the Schrödinger picture, operators are assumed to be constant and the states evolve according to the Schrödinger equation:

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle, \quad i\hbar\frac{d\hat{\rho}(t)}{dt} = [\hat{H}, \hat{\rho}] \qquad (3)$$

The state is found then as

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle, \quad U(t) = \exp\{\frac{1}{i\hbar}\int \hat{H}dt\}$$ is the evolution operator.

In the Heisenberg picture, evolution of the operators is described by the Heisenberg, or Hamilton's, equations. This approach is very similar to classical mechanics.

$$i\hbar\frac{d}{dt}\hat{A}(t) = [\hat{A}, \hat{H}]. \qquad (4)$$

Time evolution can be also understood as the evolution in the optical system (from element to element).

The mean value in the Schrödinger picture is $\langle \Psi(t) | \hat{A}_0 | \Psi(t) \rangle = \langle \Psi_0 U^+(t) | \hat{A}_0 | U(t)\Psi_0 \rangle$. It will be the same as in the Heisenberg picture if

$$\hat{A}(t) = U^+(t)\hat{A}_0 U(t) = \exp\{-\frac{1}{i\hbar}\int \hat{H}dt\}\hat{A}_0 \exp\{\frac{1}{i\hbar}\int \hat{H}dt\}. \qquad (5)$$

But indeed, this relation satisfies (4).

2. Perturbation theory and photon pairs

Let us first use the Schrödinger approach. Initially, there are no signal/idler (Stokes/antistokes) photons at the input; therefore initially the corresponding modes are in the vacuum state. The evolution of the state vector is then described by the Schrödinger equation (3) with the Hamiltonian (1):

$$i\hbar \frac{d}{dt}|\Psi\rangle = \hat{H}|\Psi\rangle ;$$

$$|\Psi\rangle = e^{\frac{1}{i\hbar}\int \hat{H} dt}|\Psi\rangle_0 \qquad (6)$$

The initial state is the vacuum. The exponential in (6) can be expanded:

$$|\Psi\rangle = e^{\Gamma t a_1^+ a_2^+}|vac\rangle \approx |vac\rangle + \Gamma t a_1^+ a_2^+|vac\rangle + \frac{1}{2}\Gamma^2 t^2 a_1^{+2} a_2^{+2}|vac\rangle + ... \qquad (7)$$

First, for simplicity we assume that the interaction is weak so we only keep the first-order terms. This is called the first order of the perturbation theory. The state then describes the generation of the two-photon Fock state or an entangled state of two photons but only in superposition with the vacuum.

3. Single-mode squeezed vacuum

What happens when the interaction becomes strong?
The Hamiltonian is the same,

$$H = i\hbar\Gamma a_1^+ a_2^+ + h.c.$$

It describes both PDC and FWM.
But the perturbation theory does not work anymore. We cannot keep only the first two terms in expansion (7).

We have to write the whole state:

$$|\Psi\rangle = e^{\Gamma t a_1^+ a_2^+ + \Gamma t a_1 a_2}|vac\rangle \approx |vac\rangle + \Gamma t a_1^+ a_2^+|vac\rangle + \frac{(\Gamma t)^2}{2}[a_1^+ a_2^+ + a_1 a_2]^2|vac\rangle + ...$$

This is what is called **squeezed vacuum**. The reason why it is called so will be clear from what follows.

We are going to describe what happens due to the Hamiltonian (1) in the case of strong interaction. In fact, we are not so much interested in the state, but rather in the mean values of observables (and their variances as well). It means that we can use the Heisenberg approach, in which we will write the Heisenberg equations and get the evolution of the operators. Then, by averaging the operator expressions over the vacuum state, we will calculate everything we need. This approach is similar to the classical description; it is more intuitive than the Schrödinger picture.

Consider first the case where the modes 1 and 2 coincide (this will be the case under certain phase matching conditions):

$$H = i\hbar\Gamma a^{+2} + h.c. \qquad (8)$$

Note that this Hamiltonian is difficult to realize with FWM where the Stokes/antistokes photons are always emitted into the sidebands of the pump beam. But it can be obtained through PDC. Such a crystal with PDC under degenerate phase matching is called a degenerate optical parametric amplifier, or a phase-sensitive parametric amplifier.

We immediately see that this will result in **light with even photon numbers**,

$$|\Psi\rangle = c_0|0\rangle + c_2|2\rangle + c_4|4\rangle + ... \qquad (9)$$

Figure 1 shows the probability distribution for the photon numbers in squeezed vacuum state (9): $P(N) = |c_n|^2$. There are zeros at odd positions, and the probabilities of even photon numbers follow a geometric progression.

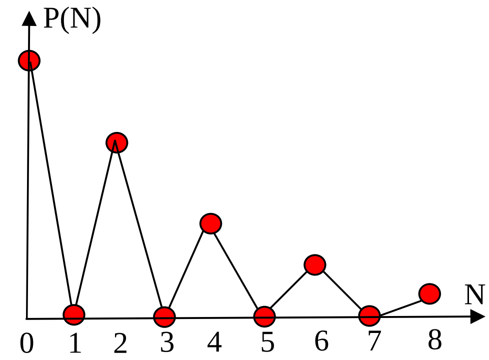

Fig.1

For calculating the evolution of the operators, we write the Heisenberg equation (4):

$$i\hbar\frac{da}{dt} = [a, H].$$

We get

$$i\hbar\frac{da}{dt} = 2i\hbar\Gamma a^+,$$

And let us pass to quadratures, $\hat{q} = (a + a^+)/2, \quad \hat{p} = (a - a^+)/2i$. For the quadratures, we get very simple equations,

$$\frac{d\hat{q}}{dt} = 2\Gamma\hat{q}, \frac{d\hat{p}}{dt} = -2\Gamma\hat{p}.$$

The solutions are $\hat{q} = e^{2\Gamma t}\hat{q}_0, \hat{p} = e^{-2\Gamma t}\hat{p}_0$. This evolution is described by hyperbolas in the phase space: $\hat{q}\hat{p} = \hat{q}_0\hat{p}_0 = const$. We can draw these hyperbolas if we assume that we are allowed to draw points in the phase space. In reality, we cannot draw points but rather, 'areas' (Fig.2).

One can see that an input coherent state (C) becomes a squeezed coherent state (SC) and is amplified or deamplified depending on its phase (that is why the term 'phase-sensitive parametric amplifier') but a vacuum (V) becomes a squeezed vacuum (SV, this is where the name comes from). Further, we will only consider SV.

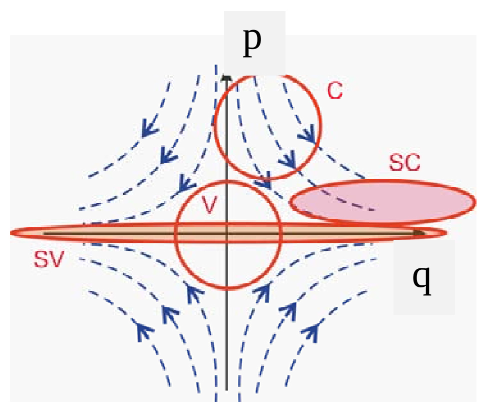

Fig.2

We see that one quadrature becomes squeezed. Why is this good? Because the noise in this quadrature is reduced at the cost of increasing the noise in the other quadrature.

We can further look into the equations; to calculate everything for this light it is reasonable to return to the photon creation and annihilation operators. Then we get the transformations

$$\hat{q} = e^{2\Gamma t}\hat{q}_0, \hat{p} = e^{-2\Gamma t}\hat{p}_0$$

$$a = \hat{q} + i\hat{p} = e^{2\Gamma t}\hat{q}_0 + ie^{-2\Gamma t}\hat{p}_0 = e^{2\Gamma t}(a_0 + a_0^+)/2 + ie^{-2\Gamma t}(a_0 - a_0^+)/2i = a_0\cosh(\Gamma t) + a_0^+\sinh(\Gamma t).$$

This equation,

$$a = a_0\cosh(\Gamma t) + a_0^+\sinh(\Gamma t), \quad (10)$$

is called the single-mode Bogolyubov transformation. The combination $\Gamma t \equiv G$ is often called the parametric gain. This parameter determines all properties of SV.

For instance, we can find the mean photon number per mode:

$$\langle N\rangle = \langle a^{+}a\rangle = \langle (a_0^{\ +}\cosh G + a_0\sinh G)(a_0\cosh G + a_0^{+}\sinh G)\rangle = \langle a_0\sinh G a_0^{+}\sinh G\rangle = \sinh^2 G.$$

At small gain, this is a linear function of the pump power, but at high gain, it becomes nonlinear.

We can also calculate the uncertainty of the quadratures. Indeed, for the variances we have

$$\Delta\hat{q}^2 = \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2 = e^{4\Gamma t}\langle\hat{q}_0^2\rangle - 0 = e^{4\Gamma t}\Delta\hat{q}_0^2,$$

$$\Delta\hat{p}^2 = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2 = e^{-4\Gamma t}\langle\hat{p}_0^2\rangle - 0 = e^{-4\Gamma t}\Delta\hat{p}_0^2.$$

Then, the squeezing of the momentum quadrature (it is also called phase quadrature, while q is called the amplitude quadrature) is given by

$$\frac{\Delta\hat{p}}{\Delta\hat{p}_0} = e^{-2\Gamma t}. \qquad (11)$$

This ratio determines the degree of squeezing and is usually given in dB.

4. Two-mode squeezed vacuum.

If there are two modes, and the Hamiltonian has the form (5), the Heisenberg equations

$$i\hbar\frac{da_1}{dt} = [a_1, H],$$

$$i\hbar\frac{da_2}{dt} = [a_2, H].$$

We obtain

$$\frac{da_1}{dt} = \Gamma a_2,$$

$$\frac{da_2}{dt} = \Gamma a_1.$$

The solutions (check) lead to the two-mode Bogolyubov transformations:

$$a_1 = a_{10}\cosh G + a_{20}^{+}\sinh G,$$

$$a_2 = a_{20}\cosh G + a_{10}^{+}\sinh G.$$

From them, we can find that not only the mean numbers in modes 1,2 are the same, $\langle N_1\rangle = \langle N_2\rangle = \sinh^2 G$, but also that the variance of the photon-number difference does not fluctuate:

$$\mathrm{Var}(\hat{N}_1 - \hat{N}_2) = \langle(\hat{N}_1 - \hat{N}_2)^2\rangle - \langle\hat{N}_1 - \hat{N}_2\rangle^2 = \langle(\hat{N}_1 - \hat{N}_2)^2\rangle = \langle(\hat{N}_1)^2\rangle + \langle(\hat{N}_2)^2\rangle - 2\langle\hat{N}_1\hat{N}_2\rangle.$$

Due to symmetry, it will be

$$\mathrm{Var}(\hat{N}_1 - \hat{N}_2) = 2\langle(\hat{N}_1)^2\rangle - 2\langle\hat{N}_1\hat{N}_2\rangle.$$

$$\langle(\hat{N}_1)^2\rangle = \left\langle\begin{matrix}(a_{10}^{\ +}\cosh G + a_{20}\sinh G)(a_{10}\cosh G + a_{20}^{+}\sinh G)\times \\ \times(a_{10}^{\ +}\cosh G + a_{20}\sinh G)(a_{10}\cosh G + a_{20}^{+}\sinh G)\end{matrix}\right\rangle =$$

$$= \langle a_{20}\sinh G(a_{10}\cosh G + a_{20}^{+}\sinh G)(a_{10}^{\ +}\cosh G + a_{20}\sinh G)a_{20}^{+}\sinh G\rangle =$$

$$= \sinh^2 G\langle(a_{10}a_{20}\cosh G + \sinh G)(a_{10}^{\ +}a_{20}^{+}\cosh G + \sinh G)\rangle = \sinh^2 G(\sinh^2 G + \cosh^2 G).$$

$$\left\langle \hat{N}_1 \hat{N}_2 \right\rangle = \left\langle \begin{matrix} (a_{10}{}^{+} \cosh G + a_{20} \sinh G)(a_{10} \cosh G + a_{20}^{+} \sinh G) \times \\ \times (a_{20}{}^{+} \cosh G + a_{10} \sinh G)(a_{20} \cosh G + a_{10}^{+} \sinh G) \end{matrix} \right\rangle =$$

$$= \left\langle a_{20} \sinh G (a_{10} \cosh G + a_{20}^{+} \sinh G)(a_{20}{}^{+} \cosh G + a_{10} \sinh G) a_{10}^{+} \sinh G \right\rangle =$$

$$= \sinh^2 G \left\langle (a_{10} a_{20} \cosh G + \sinh G)(a_{10}{}^{+} a_{20}^{+} \cosh G + \sinh G) \right\rangle = \sinh^2 G (\sinh^2 G + \cosh^2 G).$$

Therefore,

$$\mathrm{Var}(\hat{N}_1 - \hat{N}_2) = 0. \qquad (12)$$

This feature is quite unusual: imagine two independent coherent beams; then, the variance of their photon-number difference is

$$\mathrm{Var}(\hat{N}_1 - \hat{N}_2) = \mathrm{Var}(\hat{N}_1) + \mathrm{Var}(\hat{N}_2) = \left\langle \hat{N}_1 \right\rangle + \left\langle \hat{N}_2 \right\rangle,$$

because coherent beams have Poissonian statistics. This is the minimum possible in classical optics. One can introduce *noise reduction factor*,

$$NRF \equiv \frac{\mathrm{Var}(\hat{N}_1 - \hat{N}_2)}{\left\langle \hat{N}_1 + \hat{N}_2 \right\rangle},$$

and whenever it is <1, it is a signature of nonclassical behavior – it is called sub-shot-noise correlations.

Home task: Calculate the mean photon number and the degree of squeezing for the radiation at the output of a degenerate parametric amplifier with the gain (a) 1 and (b) 5.

Books:

1. Mandel, Wolf, Optical coherence and quantum optics.