Lecture 5. Instruments of quantum optics.

Quantum states. Glauber’s correlation functions. Wigner function and homodyne detection.

1. Quantum states.

In the last lecture, we obtained the quantum states generated through quadratic and cubic nonlinear interactions, for the cases of low and high strength. But let us discuss now what other states of light can exist in quantum optics.

Operators and states. In quantum mechanics, states are considered as eigenstates (eigenvectors) of various operators. For instance, we already spoke of the photon-number operator $\hat{N} = \hat{a}^+\hat{a}$. Clearly, it is a Hermitian operator, $\hat{N}^+ = \hat{N}$.

Any Hermitian operator has real eigenvalues (real *spectrum of eigenvalues*), forming a complete orthonormal set, and it corresponds to a real (measurable) observable. Accordingly, $\hat{N}$ has real eigenvalues and eigenstates:

$$\hat{N}|N\rangle = N|N\rangle. \qquad (1)$$

Its eigenstates are photon-number states, or Fock states $|N\rangle$. One can show rigorously that N is a natural number, $N = 0,1,2,...$. For these states, the number of photons is fixed. At the last lecture, we showed that parametric down-conversion and four-wave mixing, at low gain, lead to the generation of such states, although not alone (in superposition with other states).

Photon creation and annihilation operators act on Fock states as

$$a^+|N\rangle = \sqrt{N+1}|N+1\rangle,$$
$$a|N\rangle = \sqrt{N}|N-1\rangle$$

(*ladder equations*). It follows that a Fock state can be written as

$$|N\rangle = \frac{(a^+)^N}{\sqrt{N!}}|0\rangle. \qquad (2)$$

Note that photon-number states form a complete orthonormal set; accordingly, any other state can be written as an expansion over photon-number states.

Coherent states are eigenstates of photon creation and annihilation operators, which are non-Hermitian, $a^+ \neq a$. In other words,

$$a|\alpha\rangle = \alpha|\alpha\rangle,$$
$$\langle\alpha|a^+ = \alpha^*\langle\alpha|. \qquad (3)$$

The eigenvalues do not have to be real, as the operators are not Hermitian. For the same reason, the states do not necessarily form an orthonormal set. In fact, they are normalized but not orthogonal, and their set is complete, in fact, over-complete.

It is useful to write the expansion of a coherent state over Fock states (the latter form a complete set):

$$|\alpha\rangle = \sum_{N=0} c_N|N\rangle,$$

Then act on it from the left by a; we obtain

$$\alpha|\alpha\rangle = \sum_{N=0} c_N a|N\rangle = \sum_{N=0} c_N\sqrt{N}|N-1\rangle = \sum_{N'=0} c_{N'+1}\sqrt{N'+1}|N'\rangle,$$

so we obtain the recurrent relation

$$\alpha c_N = c_{N'+1}\sqrt{N'+1}, \quad c_{N+1} = \alpha\frac{c_N}{\sqrt{N+1}}.$$

From this, $c_N = \alpha^N \frac{c_0}{\sqrt{N!}}$, and c_0 will be obtained from normalization:

$$|\alpha\rangle = c_0 \sum_{N=0} \frac{\alpha^N}{\sqrt{N!}} |N\rangle,$$

$$|c_0|^{-2} = \sum_{N=0} \frac{|\alpha|^{2N}}{N!} = e^{|\alpha|^2}.$$

We finally obtain

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{N=0} \frac{\alpha^N}{\sqrt{N!}} |N\rangle. \qquad (4)$$

An example of a coherent state is the state emitted by a laser. Of course it is so only in an ideal case, where the phase of a laser is constant. In reality, the phase is drifting. But in most experiments, only the relative phase matters, and when two laser beams are to be used in the same experiment, they are usually taken from the same laser.

The vacuum state is part of both coherent and Fock sets: $|0\rangle$.

Thermal state. This is a mixed state, in contrast to coherent and Fock states, which are pure. A mixed state is simply a classical mixture of pure states. In other words, with a probability p_1 the system is in a pure state $|\psi_1\rangle$, with a probability p_2 it is in a pure state $|\psi_2\rangle$, etc. For a pure state, the density matrix is a projector: $\hat{\rho}_i = |\psi_i\rangle\langle\psi_i|$. Then, the whole density matrix is

$$\hat{\rho} = \sum_i p_i \hat{\rho}_i = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

A thermal state is described by the density matrix, which is given by a sum of projectors:

$$\rho_{th} = p_0|0\rangle\langle 0| + p_1|1\rangle\langle 1| + p_2|2\rangle\langle 2| + ...$$

This is the state, for instance, of a single mode of thermal (blackbody) radiation, and the probabilities are given by Bose-Einstein (geometric) distribution:

$$p_n \equiv \rho_{thnn} = \frac{1}{1+\langle N\rangle}\left(\frac{\langle N\rangle}{\langle N\rangle+1}\right)^n \equiv \frac{\langle N\rangle^n}{(\langle N\rangle+1)^{n+1}}. \qquad (5)$$

One also calls this state the *equilibrium state*, and the distribution *the Planck distribution*.

2. Glauber's correlation functions.

Let us recall the experiment by Hanbury Brown and Twiss, which was discussed in the introductory Lecture 1. Radiation from a thermal source (HBT used a mercury lamp or a star, in different experiments) was sent to two detectors by using a beamsplitter (in the case of a lamp) or by simply placing the two detectors apart (in the case of a star), and intensity correlation was observed. To see this intensity correlation, one can use photon-counting detectors and a coincidence scheme (cc) as shown in Fig.1, or analog detectors and a 'photocurrent multiplier'. In both cases, what is measured is the second-order correlation function.

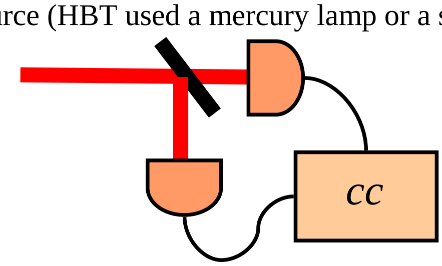

Fig.1

Second-order correlation functions. The probability of a coincidence in Fig.1, according to the theory by Roy Glauber (1963), provides the correlation function

$$G^{(2)}(t_1,t_2,r_1,r_2) \equiv \left\langle \mathrm{E}^{(-)}(t_1,r_1)\,\mathrm{E}^{(-)}(t_2,r_2)\,\mathrm{E}^{(+)}(t_1,r_1)\,\mathrm{E}^{(+)}(t_2,r_2) \right\rangle,$$

and can be written (omitting the dimensional coefficient between $E^{(-)}$ and a^+) as

$$G^{(2)}(t_1,t_2,r_1,r_2) \sim \left\langle :\hat{N}(t_1,r_1)\hat{N}(t_2,r_2): \right\rangle,$$

where the colons mean 'normal ordering' (the fact that all negative-frequency fields stand on the left and all positive-frequency fields stand on the right), and it makes sense only in quantum optics, where we have operators instead of fields. In classical optics, all variables in the angular brackets can be interchanged.

It is useful to define the normalized second-order CFs:

$$g^{(2)}(t_1,t_2,r_1,r_2) \equiv \frac{\left\langle :\hat{N}(t_1,r_1)\hat{N}(t_2,r_2): \right\rangle}{\left\langle \hat{N}(t_1,r_1)\right\rangle \left\langle \hat{N}(t_2,r_2)\right\rangle}.$$

Higher-order CFs. Similarly, higher-order intensity CFs can be defined,

$$G^{(k)}(t_1,t_2,..t_k,r_1,r_2,..r_k) \equiv \left\langle \mathrm{E}^{(-)}(t_1,r_1)\,\mathrm{E}^{(-)}(t_2,r_2)\cdots \mathrm{E}^{(-)}(t_k,r_k)\,\mathrm{E}^{(+)}(t_1,r_1)\,\mathrm{E}^{(+)}(t_2,r_2)\cdots \mathrm{E}^{(+)}(t_k,r_k) \right\rangle,$$

and their normalized analogs,

$$g^{(k)}(t_1,t_2,..t_k;r_1,r_2,..r_k) \equiv \frac{\left\langle \hat{N}(t_1,r_1)\hat{N}(t_2,r_2)\cdots \hat{N}(t_k,r_k): \right\rangle}{\left\langle \hat{N}(t_1,r_1)\right\rangle \left\langle \hat{N}(t_2,r_2)\right\rangle \cdots \left\langle \hat{N}(t_k,r_k)\right\rangle}. \qquad (6)$$

Of course in the stationary case the CFs depend only on the time differences and in the homogeneous case, only on the spatial displacements:

$$g^{(k)}(t_1,t_2,..t_k;r_1,r_2,..r_k) = g^{(k)}(t_2-t_1,..t_k-t_1;r_2-r_1,..r_k-r_1).$$

All these CFs were introduced by Glauber and are therefore called Glauber's CFs. They can be measured by adding more detectors (with beamsplitters or without them), as shown in Fig.2.

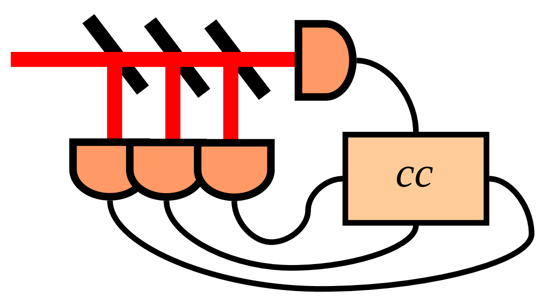

Fig.2

Second-order CFs for coherent and thermal light. In the next lecture we will consider these CFs for nonclassical light generated via various nonlinear effects. But now, for comparison, we can see what their values are for classical states, such as coherent and thermal ones. We will only focus on the values of the CFs at the center (zero space displacement and time delay),

$$g^{(k)}(t_1=t_2=...=t_k;r_1=r_2=...=r_k) \equiv g^{(k)} = \frac{\left\langle (a^+)^k a^k \right\rangle}{\left\langle a^+ a \right\rangle^k}. \qquad (7)$$

For coherent light,

$$\left\langle (a^+)^k a^k \right\rangle = \left\langle \alpha \right| (a^+)^k a^k \left| \alpha \right\rangle = |\alpha|^{2k}, \quad \left\langle a^+ a \right\rangle = |\alpha|^2,$$

hence

$$g^{(2)}_{coh} = 1.$$

For thermal light, quantum calculation is cumbersome. It gives

$$g_{th}^{(k)} = k!.$$

If the arguments of the CF in (6) do not coincide (as in (7)), other values will be observed. For instance, consider $g^{(2)}(\tau) \equiv g^{(2)}(t, t+\tau)$ in the stationary case. At very large τ, the averaging in the numerator of (6) can be done separately for each point (statistical independence), and the normalized CF is 1. This is what one can measure in a HBT experiment (Fig.3); for coherent light one gets the blue line, for thermal light, the red line, and the green line describes anti-bunched light – to be considered at the next lecture.

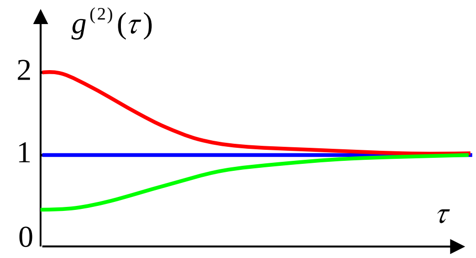

Fig.3

3. Wigner function

Photon numbers and correlation functions are part of the so-called discrete-variable quantum optics. In continuous-variable approach, in the focus are the quadrature operators,

$$\hat{q} = \frac{a^+ + a}{2}, \quad \hat{p} = \frac{a - a^+}{2i}.$$

The quantum state is described by their mean values and variances, but a more complete picture could be made by considering their joint probability distribution

$$P(\hat{q}, \hat{p}) - ?$$

We know that these operators do not commute; therefore such a joint distribution, strictly speaking, cannot be introduced. But there are many ways to introduce quasi-probabilities. As always in the probability theory, a probability density can be defined as the Fourier-transform of a certain characteristic function. In quantum optics, the characteristic function will be the mean value of some operator. Generally, there are three quasi-probabilities in quantum optics, corresponding to different ordering of a^+, a in this operator: Glauber-Sudarshan function, Wigner function, and Husimi function. But we will only consider the first two.

Glauber-Sudarshan function. A very convenient instrument in quantum optics is considering coherent states $|z\rangle \equiv |z' + iz''\rangle$ as a basis. The Glauber-Sudarshan function is introduced as the density operator in the coherent-state representation:

$$\hat{\rho} = \iint dz'\,dz'' P(z) |z\rangle\langle z|.$$

It is very convenient that in the P-representation the density matrix is diagonal. This representation has many nice properties – for instance, it provides a simple way for averaging normally-ordered operators. Indeed, consider an operator like $\hat{A} = (a^+)^n a^m$, then

$$\langle \hat{A} \rangle = Tr(\hat{A}\hat{\rho}) = \iint dz'\,dz'' P(z) \langle z|(a^+)^n a^m|z\rangle = \iint dz'\,dz'' P(z)(z^*)^n z^m. \qquad (8)$$

It means that the mean of a normally-ordered operator can be calculated in the P-representation just by putting z^* everywhere for a^+ and z for a. In this representation, it is convenient to consider the characteristic function

$$C^{(n)}(w', w'') \equiv \langle : \exp\{wa^+ - w^* a\} : \rangle \equiv \langle \exp\{wa^+\} \exp\{-w^* a\} \rangle.$$

Using (8), we obtain this mean value:

$$C^{(n)}(w', w'') = \iint dz'\,dz'' P(z) \exp\{wz^*\} \exp\{-w^* z\} = \iint dz'\,dz'' P(z) \exp\{wz^* - w^* z\}.$$

This is actually 2D Fourier transformation, and the inverse is

$$P(z', z'') = \frac{1}{\pi^2} \iint dw'\,dw'' C^{(n)}(w', w'') \exp\{-wz^* + w^* z\}.$$

This P-function cannot be really considered a probability. For a coherent state $|\alpha\rangle$, it is a delta-function:

$$P(z', z'') = \delta^{(2)}(z - \alpha).$$

And for a Fock state, it is even more singular. For this reason, it cannot be measured directly.

Wigner function. It is obtained as the Fourier-transform of a symmetric characteristic function:

$$C^{(s)}(w', w'') \equiv \langle \exp\{wa^+ - w^* a\}\rangle,$$

$$W(z', z'') = \frac{1}{\pi^2} \iint dw' dw'' C^{(s)}(w', w'') \exp\{-wz^* + w^* z\}.$$

This is the Wigner function, usually written in terms of quadratures: $W(q, p)$. It has a remarkable property: its marginal distribution gives the correct probability density for a quadrature:

$$\int dq W(q, p) = p(p).$$

For this reason, it can be measured; we will soon see how. The Wigner function can be never singular but it can be negative, which is considered as a sign of nonclassical light (impossible for a classical probability density).

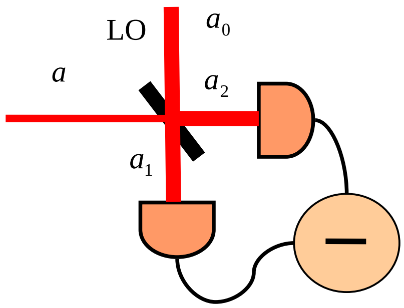

Fig.4

The Wigner function of a coherent state $|\alpha\rangle$ is a Gaussian function

$$W_{coh}(q, p) = \frac{2}{\pi} e^{-2|q+ip-\alpha|^2}.$$

It is this distribution that we were drawing at Lecture 4 for the vacuum, a coherent state, and squeezed vacuum. For the vacuum,

$$W_{vac}(q, p) = \frac{2}{\pi} e^{-2|q+ip|^2}.$$

And the uncertainties will be given by the width of this distribution.

4. Homodyne detection.

The main technique for measuring quadratures is homodyne detection. Again we have a setup with the beamsplitter and two detectors, but this time no coincidence circuit: we will not multiply but subtract the signals from the detectors (Fig.4). This time the detectors are analog ones, having high quantum efficiency and producing photocurrents: an electron is released for nearly every photon. The state under study is fed into port a, and into the other port strong coherent radiation is fed (the local oscillator, LO). To find the operators (or fields) at the beamsplitter output, one needs to know the transformation it performs. It has the form (SU(2) transformation)

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} t & r \\ -r^* & t^* \end{pmatrix} \begin{pmatrix} a \\ a_0 \end{pmatrix},$$

where $t, r,$ $|t|^2 + |r|^2 = 1,$ are transmission and reflection coefficients. Note the minus sign by one of the coefficients! t, r can be real but the minus sign will be still there; it is required due to the unitarity of the transformation.

Assume that the photocurrent in each detector reflects just the flux of photons (each photon creates an electron). Also, let the BS be 50%, so $t = r = 1/\sqrt{2}$. Then, photocurrent in detector 1 is

$$\hat{i}_1 = a_1^+ a_1 = \frac{1}{2}(a^+ + a_0^+)(a + a_0) = \frac{1}{2}\{a^+ a + a_0^+ a_0 + (a_0^+ a + h.c.)\},$$

and in detector 2,

$$\hat{i}_2 = a_2^+ a_2 = \frac{1}{2}(-a^+ + a_0^+)(-a + a_0) = \frac{1}{2}\{a^+ a + a_0^+ a_0 - (tr^* a_0^+ a + h.c.)\}.$$

The difference of these two photocurrents will be

$$\hat{i}_- \equiv \hat{i}_1 - \hat{i}_2 = a_0^+ a + h.c.$$

Assume now that the LO is so strong that its state can be considered classical. Then, the operator is replaced by a complex number $\alpha_0 = |\alpha_0| e^{i\phi}$, which is the amplitude of the LO, and

$$\hat{i}_- = |\alpha_0|(a^+ e^{i\phi} + h.c.),$$

Or, expressing the exponential through sines and cosines,

$$\hat{i}_- = |\alpha_0|[a^+ \cos\phi + ia^+ \sin\phi + a\cos\phi - ia\sin\phi] = 2|\alpha_0|[\hat{q}\cos\phi + \hat{p}\sin\phi].$$

The expression in brackets represents the ‚generalized quadrature',

$$\hat{q}(\phi) = [\hat{q}\cos\phi + \hat{p}\sin\phi]. \qquad (9)$$

Depending on the phase of the LO, it can be the first or the second quadrature.

Variance measurement. Homodyne detection is often used to observe squeezing. Then, the main observable is the variance of the quadrature, Δq^2, which is compared to the coherent-light variance. If squeezing has to be measured, it is very important that the detection has little losses.

Quantum tomography allows one, by measuring a set of quadratures (9) for different phases ϕ, reconstruct the Wigner function.

Home task: Calculate the second-order normalized CF for (a) a single-photon Fock state and (b) its superposition with the vacuum.

Books:

1. Bachor, Ralph, A guide to Experiments in Quantum Optics.