Lecture 6. Nonclassicality.

Nonclassicality: definition, experimental witnessing (anti-bunching, photon-number probability distribution, sub-shot-noise statistics, Wigner function negativity, Cauchy-Schwarz inequality).

1. Definition of nonclassicality.

Strictly, a state of light is called nonclassical if its Glauber-Sudarshan function (P-function) is negative or singular. For instance, the P-function for a coherent state is a delta-function, $P_{|\alpha\rangle} = \delta^{(2)}(z-\alpha)$: it is positive, it has a singularity, but it is integrable. In a sense, the coherent state is at the boundary between 'classical' and 'quantum'. A Fock state $|N\rangle$ has a much 'worse' P-function, $P_{|N\rangle} \sim \dfrac{d^{2N}}{d(|z|)^{2N}}\delta(|z|)$.

At the same time, it is impossible to measure a function that is singular, so something else should be measured in the experiment. There are various ways to choose such 'measurable signs' (sufficient conditions) of nonclassicality. In each case, observation of nonclassicality means that the P-function is singular or negative. It is not true the other way round (it is not a necessary condition, just a sufficient one).

An example of a value to be measured is another quasi-probability. The Wigner function already has no singularity, and the Husimi function loses both the singularity and the negativity – so they can be measured.

2. Squeezing below the shot-noise limit.

The shot-noise limit is one of the most important notions in quantum optics. It is always associated with coherent light. The word comes from the 'discrete' noise heard when shot is made – liquid drops of metal hitting the water. The phenomenon is related to the fact that light consists of photons.

For a coherent state $|\alpha\rangle$, the mean photon number is $\langle N_{coh}\rangle = \langle\alpha|a^+a|\alpha\rangle = |\alpha|^2$, and the variance is

$$\Delta N_{coh}^2 = \langle\alpha|a^+aa^+a|\alpha\rangle - \langle\alpha|a^+a|\alpha\rangle^2 = \langle\alpha|(a^+)^2a^2|\alpha\rangle + \langle\alpha|a^+a|\alpha\rangle - \langle\alpha|a^+a|\alpha\rangle^2 =$$
$$= |\alpha|^4 + |\alpha|^2 - |\alpha|^4 = |\alpha|^2 = \langle N_{coh}\rangle.$$

We see that for a coherent state, the variance is equal to the mean, or the standard deviation of the number of photons is

$$\Delta N_{coh} = \sqrt{\langle N_{coh}\rangle}.$$

This means that for coherent light, the photon number has a Poissonian distribution. This also becomes clear if we recall how a coherent state is written in terms of Fock states:

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{N=0}\frac{\alpha^N}{\sqrt{N!}}|N\rangle.$$

From this, the probability of having N photons is

$$P_{|\alpha\rangle}(N) = e^{-|\alpha|^2}\frac{|\alpha|^{2N}}{N!}. \qquad (1)$$

This is a Poissonian distribution with the mean value $\langle N\rangle = |\alpha|^2$. Any state that is less noisy is considered to be nonclassical.

Quadrature squeezing. For instance, one can measure the Wigner function or even the Husimi function and find the uncertainty (noise) of the quadratures. If for some quadrature q_φ the uncertainty is less than for the coherent state,

$$\Delta q_\varphi < \frac{1}{2}, \qquad (2)$$

then the state is certified to be nonclassical (squeezed below the shot-noise limit).

2. Sub-Poissonian statistics and anti-bunching.

Even simpler than measurement of the quadrature (homodyne measurement) is direct detection, measurement of the intensity/photon number. This way one can measure the mean number and the variance of the number of photons. The ratio of the variance to the mean is called the Fano factor,

$$F = \frac{\Delta N^2}{\langle N \rangle}. \qquad (3)$$

Whenever F<1, one says that light has sub-Poissonian statistics. Noteworthy, the Fano factor is in one-to-one correspondence with the normalized second-order correlation function (bunching parameter) $g^{(2)} = \frac{\langle :N^2: \rangle}{\langle N \rangle^2}$. Indeed,

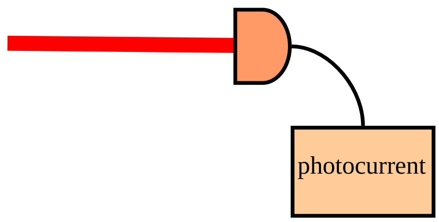

Fig.1

$$g^{(2)} - 1 = \frac{\langle :N^2: \rangle - \langle N \rangle^2}{\langle N \rangle^2} = \frac{\langle N^2 \rangle - \langle N \rangle - \langle N \rangle^2}{\langle N \rangle^2} = \frac{\Delta N^2 - \langle N \rangle}{\langle N \rangle^2} = \frac{F-1}{\langle N \rangle}. \qquad (4)$$

Whenever one observes $g^{(2)} < 1$, one speaks of *anti-bunching*, and it occurs simultaneously with sub-Poissonian statistics.

However, it depends on the state whether it is easier to measure anti-bunching or sub-Poissonian behavior. For instance, if $\langle N \rangle$ in (4) is large, then, even if F is considerably smaller than 1, $g^{(2)} - 1$ is a very small number, and it often cannot be noticed on the background of experimental noise. In this case, the correct strategy is to measure the variance of the photon number through direct detection (Fig.1). It is then important to have high quantum efficiency, low losses. The reason is that from (4),

$$F - 1 = \langle N \rangle [g^{(2)} - 1],$$

and $g^{(2)}$ is invariant to losses (see the previous lecture). Therefore, by introducing losses one only reduces $\langle N \rangle$ and hence $F-1$.

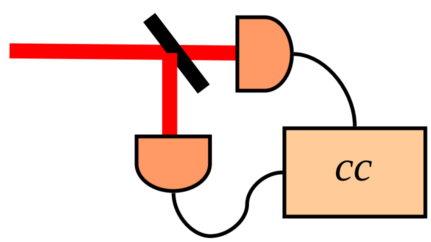

Fig.2

In the second case, when $\langle N \rangle << 1$, one should measure the correlation function in a HBT interferometer (Fig.2). In this case, losses or low quantum efficiency are not important, again because normalized CFs are invariant to losses.

One can also formulate *higher-order analogues of anti-bunching*, in terms of higher-order correlation functions: if

$$\frac{g^{(k-1)}g^{(k+1)}}{[g^{(k)}]^2} < 1, \qquad (5)$$

then light is nonclassical. This condition is also invariant to losses. In fact, anti-bunching is its particular case for k=1.

Nonclassicality can be also witnessed by measuring the photon-number probability distribution. The sufficient condition is then

$$\frac{m+1}{m}\frac{p(m+1)p(m-1)}{[p(m)]^2} < 1, \qquad (6)$$

where $p(m)$ is the probability to register m photons during a certain time.

3. Wigner function negativity.

Probably the strongest criterion that can be observed in experiment is the negativity of the Wigner function. Indeed, from $W(q_0, p_0) = 0$ at some point it follows that the P-function has a negativity/singularity. However, to measure this negativity, one has to perform the quantum tomography. This is done by measuring the probability distributions for 'all' (many) quadratures q_φ, $p_\varphi(q)$, called tomograms. Because (see Lecture 5)

$$\int dq_{\bar{\varphi}} W(q_\varphi, q_{\bar{\varphi}}) = p(q_\varphi), \qquad (7)$$

from the set of tomograms one can restore the Wigner function through the inverse Radon transform. Each tomogram is similar to the 'shadow' of the Wigner function; taken separately, each tomogram will not have any negativity, but after the reconstruction the WF will show a negativity at the center.

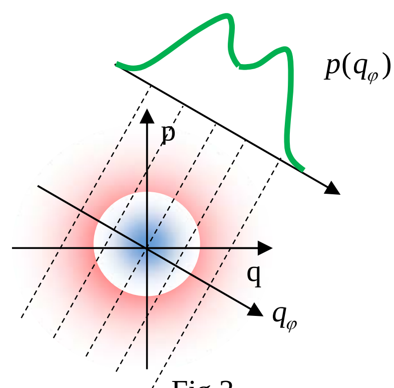

Fig.3

As an example, we can consider the tomography of a single-photon state $|1\rangle$. Its Wigner function has a negative peak at the origin and a positive ring around (Fig. 3). The tomograms will all have a 'double-peak' structure, and from the whole set one restores the Wigner function.

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4. Bipartite nonclassical features: sub-shot-noise correlations, Cauchy-Schwarz inequality.

In some cases, nonclassical features are manifested by two light beams (modes). Here we will discuss nonclassical photon-number correlations.

Imagine two independent beams hitting two different detectore (Fig.4), and let us measure the variance of their photon-number difference, $Var(N_1 - N_2)$. Because the beams are independent,

$$Var(N_1 - N_2) = Var(N_1) + Var(N_2) \geq \langle N_1 + N_2 \rangle. \qquad (8)$$

The equality is in the case where the beams are Poissonian. Therefore, it is convenient to introduce a measure, noise reduction factor (NRF):

$$NRF \equiv \frac{Var(N_1 - N_2)}{\langle N_1 + N_2 \rangle}. \qquad (9)$$

Whenever NRF<1, one speaks of sub-shot-noise correlations between the two beams.

The fact that NRF>1 follows from the Cauchy-Schwarz inequality, which, in its turn, follows from the existence of the P-function for two modes, $P(z_1, z_2)$:

$$\left\langle a_1^+ a_2^+ a_1 a_2 \right\rangle = \int d^2 z_1 d^2 z_2 P(z_1, z_2) |z_1|^2 |z_2|^2 \leq$$
$$\leq \sqrt{\int d^2 z_1 d^2 z_2 P(z_1, z_2) |z_1|^4 \int d^2 z_1 d^2 z_2 P(z_1, z_2) |z_2|^4} = \qquad (10)$$
$$= \sqrt{\left\langle (a_1^+)^2 a_1^2 \right\rangle \left\langle (a_2^+)^2 a_2^2 \right\rangle}.$$

This is the Cauchy-Schwarz inequality. After normalization, we obtain its other form:

$$g_{11}^{(2)} g_{22}^{(2)} \geq [g_{12}^{(2)}]^2. \qquad (11)$$

One can also write a weaker inequality:

$$(g_{11}^{(2)} + g_{22}^{(2)})/2 \geq \sqrt{g_{11}^{(2)} g_{22}^{(2)}} \geq g_{12}^{(2)}. \qquad (12)$$

Then, for $Var(N_1 - N_2)$ we get

$$Var(N_1 - N_2) = Var(N_1) + Var(N_2) - 2Cov(N_1, N_2) = \left\langle a_1^+ a_1 a_1^+ a_1 \right\rangle - \left\langle a_1^+ a_1 \right\rangle^2 +$$
$$+ \left\langle a_2^+ a_2 a_2^+ a_2 \right\rangle - \left\langle a_2^+ a_2 \right\rangle^2 - 2\left\langle a_1^+ a_1 a_2^+ a_2 \right\rangle + 2\left\langle a_1^+ a_1 \right\rangle \left\langle a_2^+ a_2 \right\rangle = \left\langle (a_1^+)^2 a_1^2 \right\rangle + \left\langle a_1^+ a_1 \right\rangle - \left\langle a_1^+ a_1 \right\rangle^2 +$$
$$+ \left\langle (a_2^+)^2 a_2^2 \right\rangle + \left\langle a_2^+ a_2 \right\rangle - \left\langle a_2^+ a_2 \right\rangle^2 - 2\left\langle a_1^+ a_1 a_2^+ a_2 \right\rangle + 2\left\langle a_1^+ a_1 \right\rangle \left\langle a_2^+ a_2 \right\rangle \geq 2\sqrt{\left\langle (a_1^+)^2 a_1^2 \right\rangle \left\langle (a_2^+)^2 a_2^2 \right\rangle} +$$
$$+ \left\langle a_1^+ a_1 \right\rangle + \left\langle a_2^+ a_2 \right\rangle - (\left\langle a_1^+ a_1 \right\rangle - \left\langle a_2^+ a_2 \right\rangle)^2 - 2\left\langle a_1^+ a_1 a_2^+ a_2 \right\rangle \geq \left\langle a_1^+ a_1 \right\rangle + \left\langle a_2^+ a_2 \right\rangle - (\left\langle a_1^+ a_1 \right\rangle - \left\langle a_2^+ a_2 \right\rangle)^2.$$

For beams with the same mean photon numbers, the last term disappears and we get (8).

Home task: Check conditions (5) and (6) for a coherent state.

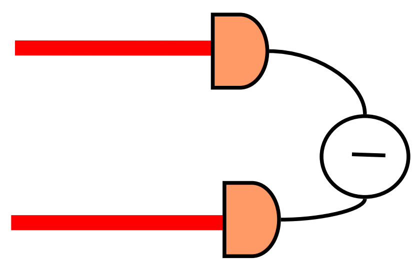

Fig.4

Books:

1. Bachor, Ralph, A Guide to Experiments in Quantum Optics.
2. Walls, Milburn, Quantum Optics.