Lecture 9. Entanglement.

Types of entanglement: frequency, wavevector, polarization. Witnesses of entanglement. Measures of entanglement. Photon-number entanglement.

1. *The EPR paradox and entangled states*

The whole subject of quantum entanglement emerged after the discussion that took place at the early time of quantum mechanics, about 80 years ago.

Continuous variables and an example from SPDC. In their paper of 1935, Einstein, Podolsky and Rosen (EPR), in the discussion with Bohr, considered an example that, from their viewpoint, showed that the quantum mechanical description was incomplete. Namely, they considered a pair of particles A,B created at some point at the same time moment, so that the conservation of momentum led to equations for their coordinates and momenta:

$$x_A - x_B = const, \quad p_A + p_B = const.$$

In other words, the positions of the particles are correlated and the momentums, anti-correlated. Then, after the particles have separated by a large distance, one can perform measurements on one of them, and hence bring it into a state with definite x or p. How then, asked EPR, the other particle is brought into a state with fixed x and p without any action on it? Does the reduction of its wavefunction occur non-locally? This led to the conclusion that the description with the wavefunction is incomplete. Moreover, one can measure the coordinate for one particle and the momentum for the other one and then violate the uncertainty relation. Such particles were called entangled (*verschränkt*); mathematically, they can be defined by the condition that the total wavefunction cannot be factorized:

$$|\Psi\rangle \neq |\Psi_A\rangle|\Psi_B\rangle.$$

Note that in the 1930-s, this example was purely a thought experiment (*Gedankenexperiment).* But today one can prepare such a state via SPDC (Fig.1) or FWM. If the parametric gain is small, the state generated in the non-collinear regime is

$$|\Psi\rangle = |0\rangle_A|0\rangle_B + c|1\rangle_A|1\rangle_B, \quad |c| << 1.$$

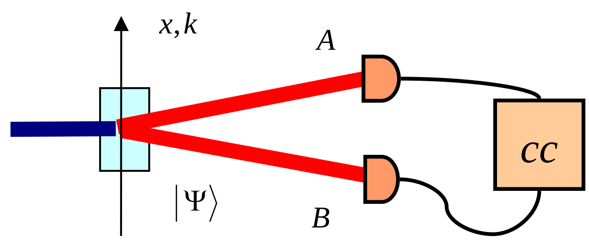

Fig.1

This is definitely an entangled state since the wavefunction is not a product of the wavefunctions of particles A, B. But this entanglement is weak (we will see it in the end). Also, this is still a simplification: there are not just two plane-wave modes A, B but a whole spectrum of k-vectors and frequencies. If, for simplicity, we fix the frequencies, then the two-photon part of the wavefunction has the form

$$|\Psi\rangle = \iint dk_A dk_B F(k_A, k_B) a^+(k_A) a^+(k_B)|0\rangle_A|0\rangle_B \ . \qquad (1)$$

(see Lecture 7), where k_A and k_B are transverse k-vectors (momenta) of the photons. If we look at the shape of the two-photon amplitude $F(k_A, k_B)$, we will see that it is stretched (Fig.2 left), i.e., the transverse wavevectors of the two photons are anti-

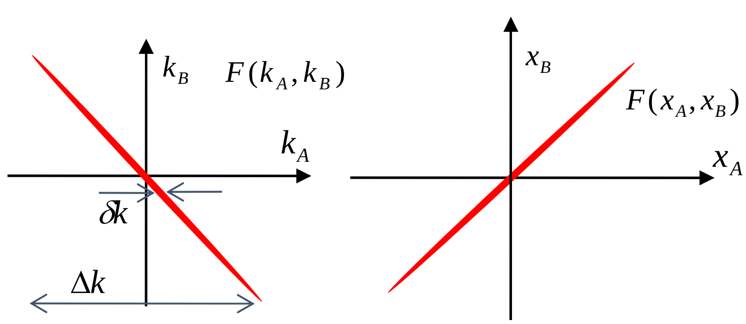

Fig.2

correlated. Note that for a photon, the k-vector is similar to the momentum: $p_{A,B} = \hbar k_{A,B}$. Similarly, the transverse coordinates of exit from the crystal for the two photons, x_A, x_B will be correlated (Fig.2a right), simply because the two photons are created 'at the same point'. More accurately, the probability amplitude in the near field $F(x_A, x_B)$ is the Fourier transform of the probability amplitude in the far field, $F(k_A, k_B)$. The state in the near field can be written as

$$|\Psi\rangle = \iint dx_A dx_B F(x_A, x_B) a^+(x_A) a^+(x_B) |0\rangle_A |0\rangle_B . \qquad (2)$$

Equations (1,2) represent an example of a state entangled in continuous variables. To measure such entanglement in experiment, one should look at both far and near field. The far-field probability distribution (Fig.2 left) is obtained by placing two detectors either simply far from the crystal (Fig.1) or into the focal plane of a lens. To obtain the near field, one should build the image of the crystal (for instance, 2F-2F scheme or, better, with a magnification) and put two detectors into the image plane.

If the photons A,B and not entangled (separable), one can obtain the condition [Mancini et al., PRL 88, 120401 (2002)]

$$\Delta(x_A - x_B)^2 \Delta(p_A + p_B)^2 \geq \left|\langle [x_A, p_A] \rangle\right|^2 = \hbar^2 . \qquad (3)$$

Or, in terms of wavevectors,

$$\Delta(x_A - x_B)^2 \Delta(k_A + k_B)^2 \geq 1 . \qquad (3‘)$$

Condition (3) is similar to the condition

$$\Delta x_B^2\big|_A \Delta p_B^2\big|_A \geq \hbar^2 , \qquad (4)$$

where $\Delta x_B^2\big|_A$ and $\Delta p_B^2\big|_A$ are variances of the conditional distribution for photon B provided that photon A was registered at some point of the far (near) field.
Condition (4) was tested in the work J. C. Howell et al., PRL 92, 210403 (2004). They found a value $0.01\hbar^2$ for (4), i.e., violated the separability condition.

Frequency and time entanglement. Similarly, SPDC generates photon pairs that are entangled in other two complementary variables, frequency and time. This is especially relevant for FWM in fibers, where there is only a single spatial mode and therefore wavevector/position entanglement cannot be observed. The state, again, can be written as

$$|\Psi\rangle = \iint d\omega_A d\omega_B F(\omega_A, \omega_B) a^+(\omega_A) a^+(\omega_B) |0\rangle_A |0\rangle_B . \qquad (5)$$

In other words, each of the photons A,B can be emitted within a broad range of frequencies, but if photon A is discovered with some frequency ω_{A0}, its match photon B will have a 'rather certain' frequency ω_{B0}. The frequencies are anti-correlated due to the condition $\omega_{A0} + \omega_{B0} = \omega_p$.
And of course there is also time entanglement,

$$|\Psi\rangle = \iint dt_A dt_B F(t_A, t_B) a^+(t_A) a^+(t_B) |0\rangle_A |0\rangle_B , \qquad (6)$$

which means that any of the photons is emitted at an uncertain time, but there is correlation between times of emission t_A and t_B.

Quadrature squeezing and entanglement. Another example of continuous-variable entanglement is quadrature entangled light, which was considered in Lecture 8. Because quadratures q, p for a light mode are similar to the position and momentum of a quantum particle, this is another example of EPR correlations. For Gaussian states, one uses the Duan criterion, which is in this case a sufficient and necessary condition for inseparability (for any states, it is a sufficient condition):

$$\Delta(q_A - q_B)^2 + \Delta(p_A + p_B)^2 \leq 2|\langle[q,p]\rangle|. \qquad (7)$$

This condition was introduced by [L.-M. Duan et al., PRL 84, 2722 (2000)] and, independently, by [R. Simon, PRL 84, 2726 (2000)].

Dichotomic variables and an example from SPDC. Much more convenient is to discuss entanglement in terms of dichotomic variables (taking only 2 values). The first formulation of the EPR paradox in terms of such variables was by Bohm who considered the projection of the spin of a spin-1/2 particle on the direction of the magnetic field. This quantity is known to be measurable in the Stern-Gerlach experiment (Fig.3, top). One can then imagine two spin-1/2 particles born in such a way that each of them has the spin direction uncertain, but there is strict correlation between the spin directions of the two particles. The corresponding wavefunction can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\{|\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B\}.$$

This state is also entangled.
The dichotomic version of the EPR paradox is important for two reasons. First, it allows for an easy derivation of an inequality (Bell's inequality) that can be tested in experiment. Second, in an experiment one can use, instead of spin ½ particles, polarized single photons. Polarization of a photon can be measured similarly to the spin direction (Fig.3, bottom), using a polarization beamsplitter instead of a large complicated setup with magnets.

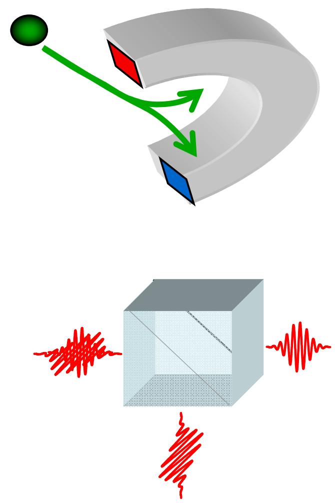

Fig.3

Polarization-entangled photons can be also produced via SPDC through type-II phase matching. See Lecture 8: the two-photon state emitted via SPDC can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\{|H\rangle_A|V\rangle_B + |V\rangle_A|H\rangle_B\}. \qquad (8)$$

This type of a state is called a Bell state, and there are four of them (see Lecture 8).

2. Measures of entanglement.

The Duan and Mancini criteria do not quantify the entanglement; like other ones (witnesses), which only tell us whether there is or there is not entanglement. More interesting are measures of entanglement, and we will discuss two of them here.

The Schmidt modes and the Schmidt number. Consider a state (1); then the two-photon amplitude (TPA) can be always represented as (the Schmidt theorem)

$$F(k_A,k_B) = \sum_{n=0}^{\infty}\sqrt{\lambda_n}\varphi_n(k_A)\psi_n(k_B), \quad \sum_{n=0}^{\infty}\lambda_n = 1. \qquad (9)$$

This is the Schmidt decomposition; $\varphi_n(k_A), \psi_n(k_B)$ are the Schmidt modes and λ_n are the Schmidt eigenvalues.

If the TPA in the form (9) is substituted in (1), the latter takes the form

$$|\Psi\rangle = \sum_{n=0}^{\infty}\sqrt{\lambda_n}\iint dk_A dk_B \varphi_n(k_A)\psi_n(k_B)a^+(k_A)a^+(k_B)|0\rangle_A|0\rangle_B =$$
$$= \sum_{n=0}^{\infty}\sqrt{\lambda_n}\int dk_A\varphi_n(k_A)a^+(k_A)|0\rangle_A\int dk_B\psi_n(k_B)a^+(k_B)|0\rangle_B = \sum_{n=0}^{\infty}\sqrt{\lambda_n}|\Psi_n\rangle_A|\Psi_n\rangle_B.$$

If there is only one term here, the state is separable; if there are many terms, the state is entangled. One can therefore quantify the degree of entanglement of a photon pair by the Schmidt number

$$K = \left[\sum_{n=0}^{\infty}\lambda_n^2\right]^{-1}. \tag{10}$$

It is large if there are many terms in the Schmidt decomposition (9). Therefore it is a measure of entanglement.

The Schmidt decomposition can be always performed numerically through the standard singular-value decomposition (svd).

The Schmidt decomposition is valid for both continuous variables and discrete variables. For instance, for state (8) the decomposition contains just two terms, with $\lambda_0 = \lambda_1 = 1/2$, and $K = 2$.

Fedorov ratio. This is an operational measure of entanglement: while the Schmidt number is not trivial to measure, the Fedorov ratio is relatively easy to measure in experiment. For instance, in the case of wavevector (k) entanglement, it is defined as

$$R = \frac{\Delta k}{\delta k}, \tag{11}$$

where Δk and δk are, respectively, the unconditional and conditional distributions of the function $|F(k_A, k_B)|^2$ (Fig.2). For a Gaussian TPA, one can show that $R = K$ exactly.

3. Twin beams entanglement.

Twin beams generated through high-gain PDC and FWM have another type of entanglement, namely entanglement in the photon number. Indeed, given the Hamiltonian

$$\hat{H} = i\hbar\Gamma a_1^+ a_2^+ + h.c.,$$

with high enough Γ, the state at the output can be written as

$$|\Psi\rangle = \sum_{N=0}^{\infty} C_N |N\rangle_1 |N\rangle_2, \tag{12}$$

which is already a Schmidt decomposition with $\lambda_N = |C_N|^2$. Therefore, the higher the gain, the slower the amplitudes $|C_N|$ decay with N, and the larger the Schmidt number. From this, it is clear that the two-photon state considered at the beginning has a very low degree of photon-number entanglement.

This kind of entangled state is macroscopic as it contains a lot of photons.

Home task: do the Schmidt decomposition for the example we considered earlier: PDC in a 3 mm long BBO crystal pumped at 800 nm with the pump beam waist 100 um. Also, calculate the Fedorov ratio, and compare R and K.

Books: Bachor, Ralph, A Guide to Experiments in Quantum Optics.