Lecture 2. Spatial and temporal coherence. Coherent modes, photon number per mode.

Spatial and temporal coherence. Measurement of the first-order CFs using interferometers. Coherence volume. Number of photons per mode. Spatial and temporal modes. Schmidt (coherent) modes.

1. *Temporal coherence.*

From the Wiener-Khinchin theorem, we see that the shape of the spectrum for any stationary radiation can be found by measuring the first-order time correlation function

$G^{(1)}(\tau) \equiv \left\langle \mathrm{E}^{(-)}(t)\,\mathrm{E}^{(+)}(t+\tau) \right\rangle.$

How to measure it? We should obviously try to split the field in two and then delay one part. Consider, for instance, doing it in the Michelson interferometer. Let the transmission of the beamsplitter be 50% (in fact it can be anything – why?).

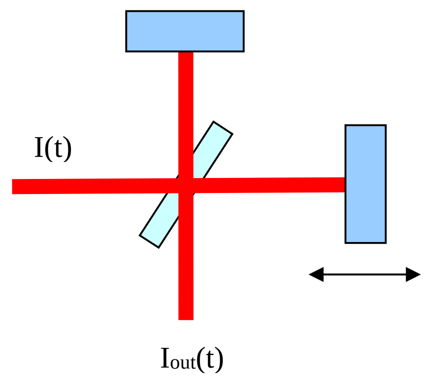

Fig.1

If the (positive-frequency) field at the input is $\mathrm{E}^{(+)}(t)$ and the path lengths are $c\Delta t$ and $c(\Delta t+\tau)$, then the field at the output will be

$\mathrm{E}_{out}^{(+)}(t) = \frac{1}{2}(\mathrm{E}^{(+)}(t-\Delta t) + \mathrm{E}^{(+)}(t-\Delta t-\tau))$. The negative-frequency field will be its complex conjugate,

$\mathrm{E}_{out}^{(-)}(t) = \frac{1}{2}(\mathrm{E}^{(-)}(t-\Delta t) + \mathrm{E}^{(-)}(t-\Delta t-\tau)).$

And the instantaneous intensity will be obtained by taking their product:

$$I_{out}(t) = \mathrm{E}_{out}^{(-)}(t)\,\mathrm{E}_{out}^{(+)}(t) = \frac{1}{4}\{I(t-\Delta t) + I(t-\Delta t-\tau) + [\mathrm{E}^{(-)}(t-\Delta t-\tau)\,\mathrm{E}^{(+)}(t-\Delta t) + c.c.]\}$$

This instantaneous intensity will fluctuate with time, so let us now average it over time (we assume the field to be ergodic).

$$\langle I_{out}\rangle = \langle I\rangle/2 + [G^{(1)}(\tau) + G^{(1)}*(\tau)]/4 = \frac{1}{2}[\langle I\rangle + \mathrm{Re}\{G^{(1)}(\tau)\}].$$

In terms of normalized CFs,

$$\langle I_{out}\rangle = \frac{\langle I\rangle}{2}[1 + \mathrm{Re}\{g^{(1)}(\tau)\}].$$

Let us introduce the slowly varying amplitude of the field:

$\mathrm{E}^{(+)}(t) = \mathrm{E}_0(t)e^{-i\omega_0 t}$, then

$G^{(1)}(\tau) \equiv \left\langle \mathrm{E}_0{}^*(t)\,\mathrm{E}_0\,(t+\tau) \right\rangle e^{-i\omega_0\tau}$, and if we change the delay (arrow in the figure), the intensity will have fast oscillations with the frequency ω_0 and the envelope. Maximum values of the output intensity will be given by

$$\langle I_{out}\rangle_{\max} = \frac{1}{2}[\langle I\rangle + |G^{(1)}(\tau)|],$$

and the minimum values, by

$$\langle I_{out}\rangle_{\min} = \frac{1}{2}[\langle I\rangle - |G^{(1)}(\tau)|].$$

Hence, the visibility, defined as $V \equiv \frac{\langle I_{out}\rangle_{\max} - \langle I_{out}\rangle_{\min}}{\langle I_{out}\rangle_{\max} + \langle I_{out}\rangle_{\min}}$, will be given by the CF modulus:

$V \equiv |g^{(1)}(\tau)|$.

Note that in this definition, it does not matter in what units we measure intensity as the visibility is dimensionless.

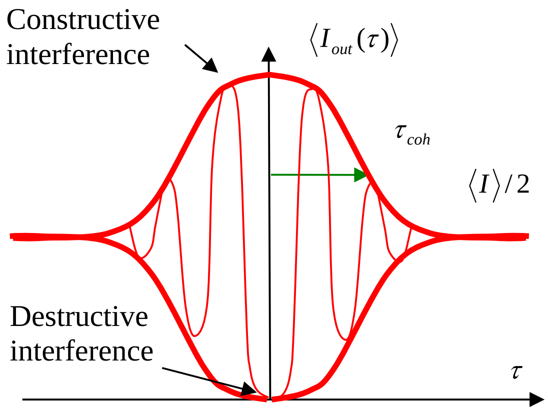

Fig.2

We see that at large delays, the interference disappears. Accordingly, at large times the CF turns to zero, $g^{(1)}(\infty)=0$. Note that at every time instance, the interference pattern (spatial, for example) actually exists but fluctuates in time. A stable interference pattern exists only at relatively small time delays τ. This is the meaning of coherence: when two fields are coherent, they interfere (form a stable interference pattern with unity visibility). Otherwise, they are partially coherent or incoherent (zero visibility of the interference pattern).

Interference visibility and the amplitudes. Note that the visibility also depends on the ratio of the amplitudes of these fields. However, one can show that this dependence is weak. Consider, for instance, two perfectly coherent fields with the amplitudes $E_{1,2}$, or intensities $I_{1,2}$. Constructive interference will give $I_{\max} = I_1 + I_2 + 2\sqrt{I_1 I_2}$, destructive interference will give $I_{\min} = I_1 + I_2 - 2\sqrt{I_1 I_2}$. The visibility is then $V = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}$. In other words, it is the ratio of the mean geometric to the mean arithmetic of the two intensities. For example, if $I_1 / I_2 = 2$, V=0.94, pretty high! For $I_1 / I_2 = 10$, V=0.57. So in the interference experiments, it is not so important to perfectly balance the two contributions.

Mach-Zehnder interferometer. Instead of Michelson, one can use other two-beam interferometers. The most common one is Mach-Zehnder (MZ), Fig.3. The bad thing about it is that scanning of one of the mirrors (changing the path) also leads to the transverse displacement of the beam. The good thing is that it also possible in a polarization version (in the same figure below). It consists of two birefringent plates. The two paths are then formed by the ordinary and extraordinary beams. Their length difference is changed by tilting the plates.

Coherence time τ_{coh}: by definition, it is the time delay at which the visibility decays twice.

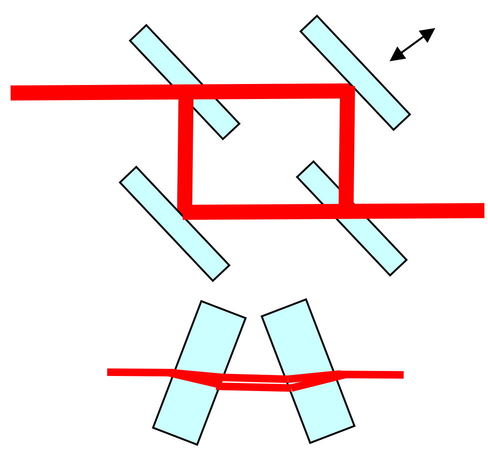

Fig.3

Strictly, it can be defined as the normalized integral of $\tau^2|G(\tau)|^2$.

Coherence length: $l_{coh} \equiv c\tau_{coh}$, the displacement of the mirror in the interferometer at which the visibility decays twice.

By virtue of the Wiener-Khinchin theorem, the width of the CF is inversely proportional to the width of the spectrum:

$$\tau_{coh} \approx \frac{\pi}{\Delta\omega};$$

Because $\omega = \frac{2\pi c}{\lambda}$, $\Delta\omega = \frac{2\pi c \Delta\lambda}{\lambda_0^{\ 2}}$, where λ_0 is the mean wavelength, and

$\tau_{coh} \approx \frac{\lambda_0^2}{2c\Delta\lambda}$, and $l_{coh} \approx \frac{\lambda_0^2}{2\Delta\lambda}$.

Examples:

1. A laser pointer: $\Delta\lambda \sim 10$ nm, $\lambda_0 \sim 600$ nm, $l_{coh} \sim 20\mu$.
2. A Ti-Sa laser: the same.
3. A good CW laser with the bandwidth 100 kHz generating at 532 nm:

$$\Delta\lambda = \frac{\Delta f \lambda_0^{\ 2}}{c} = \frac{10^5 \cdot 532^2 \cdot 10^{-18}}{3 \cdot 10^8} \frac{m^2 s}{ms} = \frac{532^2 \cdot 10^{-18}}{3 \cdot 10^5} m = 1.8 \cdot 10^{-7} nm$$

$$l_{coh} \approx \frac{532^2 \cdot 10^{-18}}{4 \cdot 10^{-16}} m \sim 700 \text{ m}.$$

2. *Spatial coherence.*

Young's experiment. To test whether fields at two spatially separated points are coherent, one should make them interfere. The natural idea is the Young interference experiment. When it was made for the first time (by Young of course), two pinholes were used and the sunlight. However, if the two pinholes were put directly into sunlight, no interference would be observed. It was necessary to have another pinhole preceding these two; otherwise the beams from the Sun hitting the two pinholes would not be coherent.

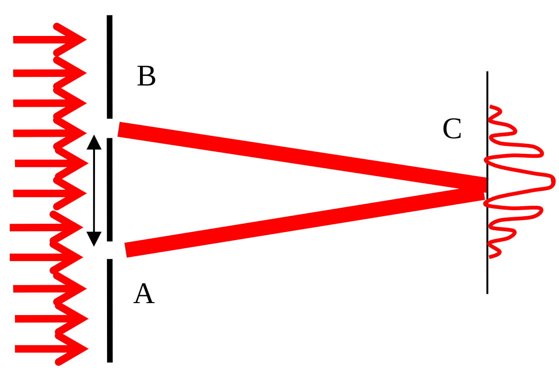

Fig.4

Measurement of the spatial CF. Indeed, let us gradually increase the separation between the pinholes and look at the interference pattern. At very small separation, the visibility will be unity but as the separation increases it will decrease.

The field at some point C is formed by the field propagating from point A and the one propagating from point B,

$$E^{(+)}(\vec{r}_C, t) = E^{(+)}(\vec{r}_A, t - \Delta t_A) + E^{(+)}(\vec{r}_B, t - \Delta t_B).$$

If point C is symmetric w.r.t. A,B, then $\Delta t_A = \Delta t_B \equiv \Delta t$. The instantaneous intensity at point C will be

$$I(\vec r_C,t)=\mathrm{E}^{(-)}(\vec r_C,t)\mathrm{E}^{(+)}(\vec r_C,t)=[\mathrm{E}^{(-)}(\vec r_A,t-\Delta t)+\mathrm{E}^{(-)}(\vec r_B,t-\Delta t)][\mathrm{E}^{(+)}(\vec r_A,t-\Delta t)+\mathrm{E}^{(+)}(\vec r_B,t-\Delta t)]=$$
$$=I(\vec r_A,t-\Delta t)+I(\vec r_B,t-\Delta t)+\{\mathrm{E}^{(-)}(\vec r_B,t-\Delta t)+c.c.\}.$$

The averaged intensity at point C will be
$$\langle I(\vec r_C,t)\rangle=\langle I(\vec r_A,t-\Delta t)\rangle+\langle I(\vec r_B,t-\Delta t)\rangle+2\mathrm{Re}G^{(1)}(\vec r_A,\vec r_B,t-\Delta t,t-\Delta t).$$
If the field is stationary and homogeneous (in the wide sense), then
$$\langle I(\vec r_C,t)\rangle=2\langle I\rangle(1+\mathrm{Re}\{g^{(1)}(\vec r_A-\vec r_B,0)\}.$$
Again, the spatial CF $g^{(1)}(\vec r_A-\vec r_B,0)\equiv g^{(1)}(\vec r_A-\vec r_B)$ will have a fast oscillating part and a slowly varying amplitude:
$$\mathrm{E}^{(+)}(\vec r)=\mathrm{E}_0(\vec r)e^{i\vec k_0\vec r},\ \text{and}\ g^{(1)}(\vec r_A-\vec r_B)=\frac{\langle E_0{}^*(\vec r_A)E_0(\vec r_B)\rangle}{\langle I\rangle}e^{i\vec k_0(\vec r_A-\vec r_B)}.$$
So the visibility of the interference will be again given by the CF modulus:
$$V\equiv|g^{(2)}(\vec r_A-\vec r_B,0)|.$$
If the points A,B are taken asymmetrically, the CF will also depend on the time,
$$V\equiv|g^{(2)}(\vec r_A-\vec r_B,\tau)|,\quad \tau\equiv\Delta t_B-\Delta t_A.$$

Coherence radius. So if we increase the distance between points A,B, the situation will be as shown in Fig.2. The distance at which the visibility decreases twice is called the coherence radius ρ_{coh}, or transverse coherence length.

The value of the coherence radius is determined by the angle at which the source (with the diameter *a*) is seen from the plane A,B:

$$\rho_{coh}\approx\theta_d z,\quad \theta_d\equiv\frac{\lambda_0}{a};$$
$$\rho_{coh}\approx\frac{\lambda_0}{a}z=\frac{\lambda_0}{\theta_a},\quad \theta_a\equiv\frac{a}{z}.$$

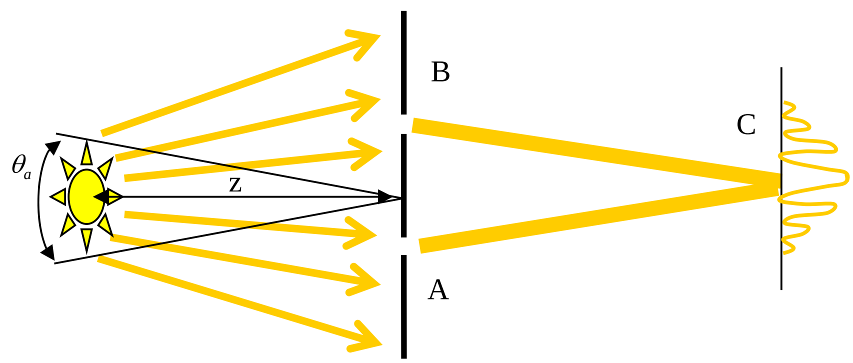

Fig.5

This relation has a simple explanation: the larger θ_a, the broader the transverse wavevector spectrum for light reaching points A,B: $\Delta k_\perp=\frac{2\pi}{\lambda_0}\theta_a$. (Remember that a is much larger than the distance between points A,B.) And the broader the transverse wavevector spectrum, the more narrow the spatial CF; this is the spatial analogue of the Wiener-Khinchin theorem. Roughly, we can write
$$\rho_{coh}\sim\frac{2\pi}{\Delta k_\perp}=\frac{\lambda_0}{\theta_a}.$$

The van Cittert-Zernike theorem. Rigorously, one should take the intensity distribution over the source

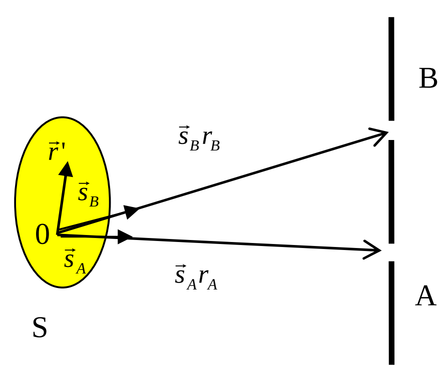

Fig.6

(which is considered as flat) and take its Fourier transform, to get the spatial CF:

$$g^{(1)}(\vec{r}_A - \vec{r}_B, 0) = \frac{\int_S I(\vec{r}\,')e^{-ik_0(\vec{s}_B - \vec{s}_A)\vec{r}\,'}d\vec{r}\,'}{\int_S I(\vec{r}\,')d\vec{r}\,'},$$

where $\vec{r}\,'$ is the coordinate on the source surface, S means the whole surface, and $\vec{s}_{A,B}$ are unit vectors from a point on the source surface towards the points $\vec{r}_{A,B}$ (Fig.6).

Michelson's stellar interferometer. We see that from the measurement of the spatial CF, one can determine the angular size of a light-emitting object. Using this principle, Michelson managed to measure the angular sizes of several bright stars. The interferometer is shown in Fig.7 (the picture taken from Wikipedia). The difficulty here is that the measurement is sensitive to the phase, hence it is vulnerable to atmospheric turbulence. It works well only for points A,B not too much separated (by not more than 10 m). So the stars should be large enough (large a) or close enough (small z). Smaller angular sizes can be measured in Hanbury Brown-Twiss stellar interferometer (Lecture 3).

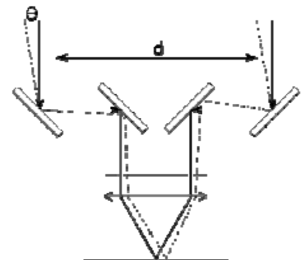

Fig.7

3. Coherence volume. Number of modes.

Coherence volume. Now, after we defined the coherence length and the coherence radius, we can imagine a part of space where radiation is coherent. It can be viewed (Fig.8) as a rectangular cylinder with the length l_{coh} and the transverse size ρ_{coh}. The volume of this cylinder is called the *coherence volume*.

$$V_{coh} = \rho_{coh}{}^2 l_{coh}$$

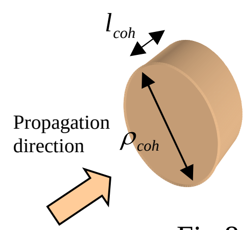

Fig.8

What is the coherence radius for sunlight? Coherence length? Coherence volume?

The same, for a laser.

So we can divide all space into 'elementary cells' as shown in Fig.8 and assume that the radiation inside each cell is coherent while the radiation in different cells is incoherent. Such a cell can be called 'a mode' (not necessarily modes should be chosen like this, but this is an option).

Detection volume. Then, one can introduce a detection volume, equal to the product of the detection time, the speed of light, and the detection area,

$$V_{\det} \equiv A_{\det} c T_{\det}.$$

And the number of detected independent modes, or independent cells of space, will be given by

$$m \equiv \frac{V_{\det}}{V_{coh}}.$$

This number should not be less than one; strictly, we should write

$$m \equiv \begin{cases} \frac{V_{\det}}{V_{coh}}, & \frac{V_{\det}}{V_{coh}} > 1 \\ 1, & \frac{V_{\det}}{V_{coh}} \le 1. \end{cases}$$

Separately, one can introduce the number of longitudinal modes and the number of transverse modes:

$$m_{\|} \equiv \frac{T_{\det}}{\tau_{coh}}, \quad m_{\perp} \equiv \frac{A_{\det}}{\rho_{coh}{}^{2}}.$$

4. Number of photons per mode (the degeneracy parameter)

An important value is the mean number of photons per coherence volume (per mode), defined by Mandel as the degeneracy parameter. It is obtained if the energy contained in the coherence volume is divided by the energy of a single photon. For instance, if a source has the mean intensity $\langle I \rangle$, then the energy flowing through the coherence area per unit time (the power) is
$P = \langle I \rangle \rho_{coh}^2$. Then, the energy in the coherence volume is obtained just by multiplying this by the coherence time,

$$W_{coh} = \langle I \rangle \rho_{coh}^2 \tau_{coh} = \langle I \rangle V_{coh} / c,$$

and the mean number of photons per mode is

$$N_{\text{mod}e} \equiv \frac{W_{coh}}{\hbar\omega} = \frac{\langle I \rangle V_{coh}}{\hbar\omega c}.$$

- It is called the degeneracy parameter because it shows the occupation number of a single ‚cell'; while for fermions this occupation number cannot exceed 1, for photons it can take any value.
- It is this number of photons per mode that is given by the Planck formula for the blackbody radiation.
- In nonlinear optics, the interaction efficiency is given by $N_{\text{mod}e}$. Hence the degeneracy parameter of a light field also determines its ability to interact with other light fields.
- The degeneracy factor is also important because it tells whether the photon structure of light is pronounced ($N_{\text{mod}e} \le 1$) or not ($N_{\text{mod}e} >> 1$).

Space/time or wavevector/frequency. Equivalently, one can define coherence volume in the space given not by Cartesian coordinates and time but by wavevector and frequency.

5. Modes: plane monochromatic waves or something else? (Coherent-mode, or Schmidt-mode, representation)

Consider a two-point spatial CF $G^{(1)}(\vec{r}_1, \vec{r}_2) \equiv G^{(1)}(\vec{r}_1, \vec{r}_2, 0)$. It is not factorable in the general case.
In the special case where it is factorable as

$$G^{(1)}(\vec{r}_1, \vec{r}_2) = f^*(\vec{r}_1) f(\vec{r}_2),$$

the normalized CF is unimodular:

$$\left| g^{(1)}(\vec{r}_1, \vec{r}_2) \right| \equiv \frac{\left| G^{(1)}(\vec{r}_1, \vec{r}_2) \right|}{\sqrt{G^{(1)}(\vec{r}_1, \vec{r}_1) G^{(1)}(\vec{r}_2, \vec{r}_2)}} = \frac{\left| f^*(\vec{r}_1) f(\vec{r}_2) \right|}{\sqrt{f^*(\vec{r}_1) f(\vec{r}_1) f^*(\vec{r}_2) f(\vec{r}_2)}} = 1.$$

It means that such a field is spatially coherent everywhere.

But even in the general case the CF can be represented as a sum of such coherent terms. Indeed, according to the so-called Mercer's theorem [Mandel&Wolf], for any 'good' function of two variables (in our case, $G^{(1)}(\vec{r}_2,\vec{r}_1) = G^{(1)^*}(\vec{r}_1,\vec{r}_2)$) there exists the representation (called also the Mercer expansion or the Schmidt decomposition)

$$G^{(1)}(\vec{r}_1,\vec{r}_2) = \sum_n \alpha_n f_n{}^*(\vec{r}_1) f_n(\vec{r}_2),$$

where α_n, $f_n(\vec{r}_1)$ are eigenfunctions and eigenvalues of the integral equation (Fredholm equation)

$$\int_D d\vec{r}_2 G^{(1)}(\vec{r}_1,\vec{r}_2) f_n(\vec{r}_2) = \alpha_n f_n(\vec{r}_1).$$

The eigenvalues are real and positive, $\alpha_n \geq 0$,
and the eigenfunctions are orthonormal,

$$\int_D d\vec{r}\; f_n(\vec{r}\,) f_m{}^*(\vec{r}\,) = \delta_{nm}.$$

Then we see that the CF is represented as a series of factorable CFs,

$$G^{(1)}(\vec{r}_1,\vec{r}_2) = \sum_n \alpha_n G_n^{(1)}(\vec{r}_1,\vec{r}_2),$$

$$G_n^{(1)}(\vec{r}_1,\vec{r}_2) \equiv f_n{}^*(\vec{r}_1) f_n(\vec{r}_2).$$

Each of these factorable CFs is unimodular, as we have shown above. Hence, they represent *coherent modes*. And these coherent modes are not plane waves but should be found separately for each CF. For instance, in the case of a double-Gaussian CF (Schell model), $G^{(1)}(\vec{r}_1,\vec{r}_2) \sim \exp\{-(\vec{r}_1-\vec{r}_2)^2/2\sigma_-^2\}\exp\{-(\vec{r}_1+\vec{r}_2)^2/2\sigma_+^2\}$, coherent modes are given by Hermite-Gaussian polynomials.

The same consideration is valid for temporal CF.

Home task:
Find the coherence volume for a laser pointer emitting into a bandwidth of 10 nm around 650 nm. The beam has a diameter of 3 mm and a diffraction divergence (Why does it matter?) At what power will the laser pointer have a single photon in the coherence volume?

Books:

1. Mandel & Wolf, Optical coherence and quantum optics, Sec. 4.2-4.4, 4.7
2. Klyshko, Physical foundations of quantum electronics, Sec. 7.2.3-7.2.6