Lecture 5. Quantization of the electromagnetic field.

Problems with semiclassical description. Modes: monochromatic plane waves or other. Maxwell's equations. Energy of the field. Position and momentum for the field. Passing to operators. Photon creation, annihilation, and number operators.

1. *Problems with semiclassical description.*

Let us recall things we discussed at the previous lectures. Several times we mentioned: this is what the classical statistical optics predicts, but this is violated in some experiments. These were the following cases.

Sub-Poissonian behaviour. At Lecture 3 we have derived from classical description: the variance of the photocounts number should be always larger than its mean, $\Delta m^2 \geq \langle m \rangle$. It was mentioned that in experiments, this can be violated for certain types of light (amplitude squeezed light).

Anti-bunching. At Lecture 3, we mentioned anti-bunching: $g^{(2)}(0) < 1$. Again, for certain types of light, like the one emitted by single atoms/quantum dots/etc., this will be violated. Anti-bunched light will be discussed at Lecture 11.

Shot noise. At Lecture 3, we assumed that there is some kind of light (coherent), for which the intensity does not fluctuate, the intensity is constant. However, even for coherent light, there are intensity fluctuations associated to the photon (discrete) structure of light. This is called the shot noise. For instance, Fig.1 shows a way to observe it: after a beamsplitter, mean photon numbers (or mean intensities) measured by two detectors will be equal if the beam splitting is balanced, but there will be noise of their difference: $Var(N_1 - N_2) = \langle N_1 + N_2 \rangle$.

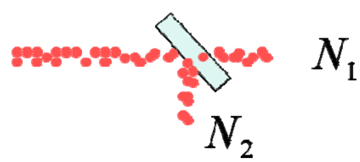

Fig.1

The shot noise cannot be derived within the classical description of light. One can argue that the shot noise might come not from light but from the detection (discrete charge etc.) but then, it is not clear how one can suppress it by specially engineering the light source (squeezed light). Squeezed light will be discussed at Lecture 10.

At Lecture 2, we discussed the photon number per mode and said that when it is large, light behaves classically. However, shot noise will be present also for very bright light, with huge photon number per mode. So, in a sense, light never behaves classically.

Spontaneous transitions in atoms. At Lecture 4, we mentioned spontaneous transitions in atoms. They cannot be derived from the semiclassical theory of atom-light interaction.

Spontaneous parametric down-conversion. Similarly to spontaneous transitions in atoms, spontaneous decay of photons in pairs cannot be described in the framework of the classical theory. But the inverse process, second harmonic generation, similarly to stimulated transitions, can be described classically. We will discuss spontaneous parametric down-conversion at Lecture 10.

2. Modes: monochromatic plane waves or other.

Modes. To pass to the quantum description of light, we have to introduce independent modes. In Lecture 2, we considered modes as plane monochromatic waves, or groups of such waves, with certain angular divergence (coherent radius) and frequency width (coherence time). We also mentioned that another choice is possible, so-called *coherent modes*. We will return to that concept later, and now, for simplicity, we will introduce a discrete set of plane-wave modes.

A real field is always limited in space and time. Fig. 1 shows how one can assume that the field only exists between $z=0$ and $z=L$. But outside of this region, we do not care, so we can extend it periodically to infinity (Fig.1). With the periodicity $E(z)=E(z+L)$, the electric field can be written as a Fourier series,

$$E(z)=\sum_{m=-\infty}^{\infty}E_m e^{ik_m z},\quad k_m=2\pi m/L,\quad m=0,\pm1,\pm2,... \tag{1}$$

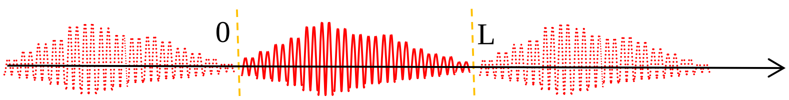

Fig.2

The spatial harmonics can be found by multiplying (1) by $e^{-ik_n z}$ and integrating from –L/2 to L/2:

$$\int_{-L/2}^{L/2}dzE(z)e^{-ik_n z}=\sum_{m=-\infty}^{\infty}E_m\int_{-L/2}^{L/2}dze^{i(k_m-k_n)z}=L\sum_{m=-\infty}^{\infty}E_m\operatorname{sinc}\frac{(k_m-k_n)L}{2}=L\sum_{m=-\infty}^{\infty}E_m\operatorname{sinc}[\pi(m-n)]=$$

$$=L\sum_{m=-\infty}^{\infty}E_m\delta_{mn}=LE_n,$$

hence

$$E_n=\frac{1}{L}\int_{-L/2}^{L/2}dzE(z)e^{-ik_n z}. \tag{2}$$

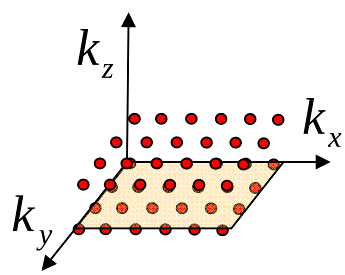

Fig.3

This can be done in all 3 dimensions, so (1) can be written in the vector form,

$$\vec{E}(\vec{r})=\sum_{l,m,n=-\infty}^{\infty}\vec{E}_{lmn}e^{i(k_l x+k_m y+k_n z)}=\sum_{\vec{k}}\vec{E}_{\vec{k}}e^{i\vec{k}\vec{r}},\quad \vec{k}=\{k_x;k_y;k_z\}=\{k_l;k_m;k_n\}=\frac{2\pi}{L}\{l,m,n\}. \tag{3}$$

Because the field is real, we have $\vec{E}_{\vec{k}}=\vec{E}^*_{-\vec{k}}$. The backward wave amplitude is complex conjugate of the forward wave amplitude.

Thus, we introduced modes as plane waves, occupying nodes in the k-space (Fig.3); the distance between these nodes is $\frac{2\pi}{L}$, and if L is large (which can always be the case), the modes are very densely packed, and summation in (3) can be replaced by integration.

Coherent modes instead of plane-wave ones. It is useful to keep in mind that the choice of the modes was arbitrary, and in future we can chose them differently, as, for instance, coherent (Mercer's) modes.

The same way, modes can be introduced for magnetic field B:

$$\vec{B}(\vec{r}) = \sum_{\vec{k}} \vec{B}_{\vec{k}} e^{i\vec{k}\vec{r}}, \quad \vec{k} = \{k_x; k_y; k_z\} = \frac{2\pi}{L}\{l,m,n\}. \tag{3'}$$

3. *Maxwell's equations.*

Recall Maxwell's equations in free space in terms of electric field $\vec{E}$ and magnetic field $\vec{B}$:

$$\vec{\nabla}\times\vec{B} - \varepsilon_0\mu_0\dot{\vec{E}} = 0; \quad \varepsilon_0\mu_0 = 1/c^2;$$
$$\vec{\nabla}\times\vec{E} + \dot{\vec{B}} = 0;$$
$$(\vec{\nabla}\cdot\vec{E}) = 0;$$
$$(\vec{\nabla}\cdot\vec{B}) = 0.$$

(Again, the dot means time differentiation.) From the two last equations, it follows that electromagnetic field in a free-space plane wave is transverse. And we proceed with the first two,

$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\dot{\vec{E}} = 0;$$
$$\vec{\nabla}\times\vec{E} + \dot{\vec{B}} = 0.$$

Then we assume (3,3') with only 2 transverse dimensions, σ denoting polarization,

$$\vec{E}(\vec{r}) = \sum_{\vec{k}} \vec{E}_{\vec{k}} e^{i\vec{k}\vec{r}}, \quad \vec{B}(\vec{r}) = \sum_{\vec{k}} \vec{B}_{\vec{k}} e^{i\vec{k}\vec{r}}, \quad \vec{k} = \{k_x; k_y; k_z; \sigma\}; \quad k_x, k_y, k_z = \frac{2\pi}{L}\{l,m,n\}. \tag{4}$$

The Maxwell equations then yield, for each mode,

$$i\vec{k}\times\vec{B}_{\vec{k}} e^{i\vec{k}\vec{r}} - \frac{1}{c^2}\dot{\vec{E}}_{\vec{k}} e^{i\vec{k}\vec{r}} = 0,$$
$$i\vec{k}\times\vec{E}_{\vec{k}} e^{i\vec{k}\vec{r}} + \dot{\vec{B}}_{\vec{k}} e^{i\vec{k}\vec{r}} = 0.$$

After eliminating the exponentials, we differentiate the first equation, multiply it by vector $\vec{k}$ and substitute into it the second one; we get

$$\ddot{\vec{E}}_{\vec{k}} + c^2k^2\vec{E}_{\vec{k}} = 0.$$

For the magnetic field, we have

$$\vec{B}_{\vec{k}} = \frac{i}{c^2k^2}\vec{k}\times\dot{\vec{E}}_{\vec{k}}. \tag{5}$$

Because $c^2k^2 = \omega_{\vec{k}}^2$ (dispersion dependence in the vacuum), we obtain

$$\ddot{\vec{E}}_{\vec{k}} + \omega_{\vec{k}}^2\vec{E}_{\vec{k}} = 0. \tag{6}$$

This means that the field in each mode $\vec{k}$ behaves like a harmonic oscillator with the frequency ω_k.

The solution to (6) has the form

$$\vec{E}_{\vec{k}}(t) = \frac{1}{2}\left(\vec{E}_{\vec{k}0}e^{-i\omega_{\vec{k}}t} + [\vec{E}_{\vec{k}0}]^* e^{i\omega_{\vec{k}}t}\right). \tag{7}$$

And from this, for the magnetic field we have

$$\vec{B}_{\vec{k}}(t) = \frac{1}{2c}\frac{\vec{k}}{k}\times\left(\vec{E}_{\vec{k}0}e^{-i\omega_{\vec{k}}t} - [\vec{E}_{\vec{k}0}]^* e^{i\omega_{\vec{k}}t}\right). \tag{8}$$

Because $\vec{E}_{\vec{k}}^{\,*} = \vec{E}_{-\vec{k}}$, we finally obtain

$$\begin{aligned} \vec{E}(t) &= \mathrm{Re}\sum_{\vec{k}}\vec{E}_{\vec{k}0}e^{i\vec{k}\vec{r}-i\omega_{\vec{k}}t}, \\ \vec{B}(t) &= \mathrm{Re}\sum_{\vec{k}}\frac{\vec{k}}{ck}\times\vec{E}_{\vec{k}0}e^{i\vec{k}\vec{r}-i\omega_{\vec{k}}t}. \end{aligned} \tag{9}$$

The wavevectors $\vec{k}$ here span all directions, forward and backward. It is important that for backward waves,

$$\vec{E}_{-\vec{k}} = \vec{E}_{\vec{k}}^{\,*} \text{ and } \vec{B}_{-\vec{k}} = -\vec{B}_{\vec{k}}^{\,*}. \tag{10}$$

Thus, we introduced modes, and denoted them by index $\vec{k}$, which means not only 3 numbers $\vec{k} = \frac{2\pi}{L}\{l,m,n\}$, but also polarization. A mode will thus be given by the set $\{l,m,n,\sigma\}$. Now, let us consider only one polarization and omit the vector notation of the field. We will also omit the vector index $\vec{k}$ and write simply k.

4. Energy of the electromagnetic field.

Let us find the energy of the electromagnetic field in the volume bounded by the size L. It is

$$\mathscr{E} = \frac{1}{2}\int_{L^3} d\vec{r}\,(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2). \tag{11}$$

Substituting (3) (and omitting the vectors), we get

$$\mathscr{E} = \frac{1}{2}\int_{L^3} d\vec{r}\,(\varepsilon_0\sum_k E_k(t)e^{i\vec{k}\vec{r}}\sum_{k'}E_{k'}(t)e^{i\vec{k}'\vec{r}} + \frac{1}{\mu_0}\sum_k B_k(t)e^{i\vec{k}\vec{r}}\sum_{k'}B_{k'}(t)e^{i\vec{k}'\vec{r}}).$$

Because $\int_{L^3} d\vec{r}e^{i(\vec{k}_n-\vec{k}_m)\vec{r}} = L^3\delta_{mn}$, the energy (we will further call it the Hamiltonian) takes the form

$$\mathscr{H} = \frac{L^3}{2}\sum_k(\varepsilon_0|E_k(t)|^2 + \frac{1}{\mu_0}|B_k(t)|^2) = \frac{L^3}{2}\sum_k(\varepsilon_0 E'^2_k + \varepsilon_0 E''^2_k + \frac{1}{\mu_0}B'^2_k + \frac{1}{\mu_0}B''^2_k). \tag{12}$$

5. Canonical variables (position and momentum) for the field.

It is useful to introduce, instead of the complex field harmonics $E_k(t)$ and $B_k(t)$ the so-called canonical variables (positions and momentums),

$$q_k \equiv \sqrt{\frac{L^3}{\omega_k^{\ 2}}}(\sqrt{\varepsilon_0}E_k'' + \frac{1}{\sqrt{\mu_0}}B_k''),$$
$$p_k \equiv -\sqrt{L^3}(\sqrt{\varepsilon_0}E_k' + \frac{1}{\sqrt{\mu_0}}B_k'). \tag{13}$$

We chose the coefficients in this tricky way in order to make the Hamiltonian look like the energy of the harmonic oscillator:

$$\frac{1}{2}\left[\omega_k^{\ 2}q_k^2 + p_k^2\right] = \frac{L^3}{2}(\varepsilon_0 E''^{\ 2}_k + \frac{1}{\mu_0}B''^{\ 2}_k + 2\sqrt{\frac{\varepsilon_0}{\mu_0}}E''_k B''_k + \varepsilon_0 E'^{\ 2}_k + \frac{1}{\mu_0}B'^{\ 2}_k + 2\sqrt{\frac{\varepsilon_0}{\mu_0}}E'_k B'_k) =$$
$$= \frac{L^3}{2}\left[\varepsilon_0 |E_k|^2 + \frac{1}{\mu_0}|B_k|^2 + 2\sqrt{\frac{\varepsilon_0}{\mu_0}}(E''_k B''_k + E'_k B'_k)\right].$$

The expression in the round brackets will be zero because for backward waves (index $-k$), the values will be (see (10))

$$E''_{-k} = -E''_k ; B''_{-k} = B''_k ;$$
$$E'_{-k} = E'_k ; B'_{-k} = -B'_k .$$

Then, we get the Hamiltonian in the form

$$\mathscr{H} = \frac{1}{2}\sum_k (\omega_k^{\ 2} q_k^2 + p_k^2). \tag{14}$$

Thus, each mode k of the light field is described as a harmonic oscillator with the frequency ω_k, position q_k and momentum p_k.

Fig.4

And one can check (we will not) that Hamilton's equations hold true,

$$\dot{q}_k = \partial\mathscr{H}/\partial p_k ; \quad \dot{p}_k = -\partial\mathscr{H}/\partial q_k . \tag{15}$$

It is also convenient to introduce a single complex value instead of these two:

$$a_k \equiv \sqrt{\frac{1}{2\hbar\omega_k}}(\omega_k q_k + i p_k). \tag{16}$$

Then, the Hamiltonian can be written as

$$\mathscr{H} = \sum_k \hbar\omega_k |a_k|^2 . \tag{17}$$

These new variables a_k can be written in terms of the electric field as

$$a_k \equiv -i\sqrt{\frac{L^3}{2\hbar\omega_k}}\left[\sqrt{\varepsilon_0}E_k + \frac{1}{\sqrt{\mu_0}}B_k\right].$$

5. Quantization of the field.

Now, that each mode of the electromagnetic field is described as a harmonic oscillator, and some quantities have the physical meaning of the position and momentum of this mode, *the quantization* consists of assigning *operators* to these quantities. We will mark this by simply writing hats over the letters:

$$\hat{q}_k ; \quad \hat{p}_k .$$

But in quantum mechanics, the operators, in the general case, do not commute. For instance, the commutator of position and momentum is

$$[\hat{q}_k, \hat{p}_n] \equiv \hat{q}_k \hat{p}_n - \hat{p}_n \hat{q}_k = i\hbar\delta_{kn}. \quad (18)$$

The position and momentum operators are Hermitian ones, $\hat{q}_k^+ = \hat{q}_k, \quad \hat{p}_k^+ = \hat{p}_k$, because they were formed from real variables, hence they will have real eigenvalues.

The Hamiltonian then also becomes an operator, a Hermitian one, and takes the form

$$\hat{H} = \frac{1}{2}\sum_k (\omega_k{}^2 \hat{q}_k^2 + \hat{p}_k^2). \quad (19)$$

From (16), we obtain the non-Hermitian operator

$$\hat{a}_k \equiv \sqrt{\frac{1}{2\hbar\omega_k}}(\omega_k \hat{q}_k + i\hat{p}_k),$$

and its Hermitian conjugate,

$$\hat{a}_k^+ \equiv \sqrt{\frac{1}{2\hbar\omega_k}}(\omega_k \hat{q}_k - i\hat{p}_k).$$

They are called photon creation and annihilation operators, for the reasons that will be clear later. Their commutator is

$$[\hat{a}_k, \hat{a}_k^+] \equiv \frac{1}{2\hbar\omega_k}[\omega_k \hat{q}_k + i\hat{p}_k, \omega_k \hat{q}_k - i\hat{p}_k] = 1.$$

Also, it is useful to calculate

$$\hat{a}_k^+ \hat{a}_k \equiv \frac{1}{2\hbar\omega_k}(\omega_k \hat{q}_k - i\hat{p}_k)(\omega_k \hat{q}_k + i\hat{p}_k) = \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 + i\omega_k[\hat{q}_k, \hat{p}_k]) =$$

$$= \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2 - \hbar\omega_k) = \frac{1}{2\hbar\omega_k}(\omega_k^2 \hat{q}_k^2 + \hat{p}_k^2) - \frac{1}{2}.$$

Then, the Hamiltonian is

$$\hat{H} = \sum_k \hbar\omega_k (\hat{a}_k^+ \hat{a}_k + \frac{1}{2}). \quad (20)$$

The operator $\hat{N} \equiv \hat{a}_k^+ \hat{a}_k$ is called the photon-number operator; one can say that it shows how much the mode k is populated with photons.

Comparing Hamiltonian (20) to its classical version (17), we see that, in addition to the mode contributions, there is also a background, which is infinite, if infinite is the number of modes. This is called zero-point vacuum fluctuations (even if a mode is not populated it still has some energy) and it is sometimes considered as a serious problem of field quantization.

Home task:
What is the ratio of the electric and magnetic energies in one field mode?

Books:

1. Mandel & Wolf, Optical coherence and quantum optics
2. Klyshko, Physical foundations of quantum electronics