Wintersemester 2015/16, Universität Erlangen-Nürnberg, Prof. Dr. Florian Marquardt

Blatt 2 - Lösung

Aufgabe 1: Zirkular polarisierte Welle

The electric field of an electromagnetic wave propagating in the z direction is $\vec{E}(\vec{r},t) = \mathrm{Re}[\vec{E}_0 e^{i(kz-\omega t)}]$, where $\vec{E}_0$ is a complex amplitude vector. Since $E_{0x} = E$ and $E_{0y} = iE$, the complex vector is $\vec{E}_0 = \begin{pmatrix} E \\ iE \\ 0 \end{pmatrix}$. It means that

$$E_x(\vec{r},t) = \mathrm{Re}[Ee^{i(kz-\omega t)}] = E\cos(kz-\omega t)\,,$$

$$E_y(\vec{r},t) = \mathrm{Re}[iEe^{i(kz-\omega t)}] = \mathrm{Re}[Ee^{i(kz-\omega t+\frac{\pi}{2})}] = E\cos\left(kz-\omega t+\frac{\pi}{2}\right) = -E\sin(kz-\omega t)\,,$$

$$E_z(\vec{r},t) = 0,$$

so the electric field is circularly polarised (i.e. the phase difference is 90 degrees and the amplitudes are equal). Furthermore we know from the Maxwell equations that the relationship between the electric and magnetic fields of an electromagnetic wave is

$$\vec{B}(\vec{r},t) = \frac{1}{c}\frac{\vec{k}}{\mid\vec{k}\mid}\times\vec{E}(\vec{r},t) = \frac{1}{c}\mathrm{Re}[\begin{pmatrix} -iE \\ E \\ 0 \end{pmatrix} e^{i(kz-\omega t)}] = \frac{1}{c}\mathrm{Re}[-i\vec{E}_0 e^{i(kz-\omega t)}].$$

And since the multiplication with $-i = \exp\left(-\frac{\pi}{2}\right)$ is equivalent with -90 degrees phase shift, the resulting magnetic field components are

$$B_x(\vec{r},t) = \frac{E}{c}\sin(kz-\omega t)\,,$$

$$B_y(\vec{r},t) = \frac{E}{c}\cos(kz-\omega t)\,,$$

$$B_z(\vec{r},t) = 0,$$

Using these result we can easily plot the electric and magnetic field components. With fixed time both fields form a spiral around the z axis and the two fields are perpendicular at each point. The electric and magnetic field distributions are shown in Fig. 1.

The electromagnetic energy density is

$$\mathcal{E} = \frac{1}{2}\epsilon_0 \mid\vec{E}\mid^2 + \frac{1}{2}\frac{1}{\mu_0}\mid\vec{B}\mid^2\,.$$

Since $\mid\vec{B}\mid = \mid\vec{E}\mid/c$ and $\mid\vec{E}\mid^2 = E_x^2 + E_y^2 = E^2\left(\cos^2(kz-\omega t) + \sin^2(kz-\omega t)\right) = E^2$, the energy density is

$$\mathcal{E} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\frac{1}{\mu_0 c^2}E^2 = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\epsilon_0 E^2 = \epsilon_0 E^2,$$

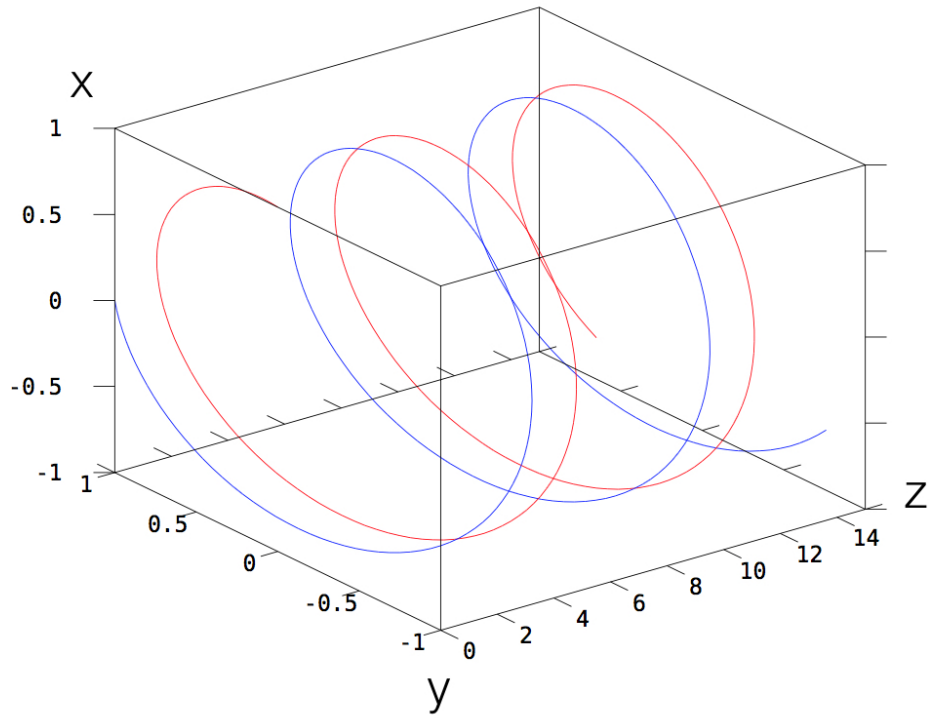

Figure 1: In this figure the schematic illustration of the electric (red line) and magnetic (blue line) fields of the circularly polarised electromagnetic wave propagating in the z direction is presented.

where we used the $c^2 = (\epsilon_0\mu_0)^{-1}$ identity. It is generally (i.e. not only in vacuum) true that the electromagnetic energy carried by an electromagnetic wave is equally distributed between the electric field energy and the magnetic field energy parts.

To calculate the Poynting vector $\vec{S}(\vec{r},t)$ we can use its definition

$$\vec{S}(\vec{r},t) = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\begin{pmatrix} E\cos(kz-\omega t) \\ -E\sin(kz-\omega t) \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{E}{c}\sin(kz-\omega t) \\ \frac{E}{c}\cos(kz-\omega t) \\ 0 \end{pmatrix} = \frac{1}{\mu_0}\frac{E^2}{c}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\mu_0 c}E^2\hat{k},$$

and using again the $c^2 = (\epsilon_0\mu_0)^{-1}$ identity we get

$$\vec{S}(\vec{r},t) = c\epsilon_0 E^2\hat{k} = c\mathcal{E}\hat{k}.$$

It means that for circularly polarised plane wave the energy density and the Poynting vector are constant in time and space.

If we now consider linear polarisation with $E_x(\vec{r},t) = E\cos(kz-\omega t)$ and $E_y(\vec{r},t) = 0$, then the energy density is

$$\mathcal{E} = \frac{1}{2}\epsilon_0 \mid \vec{E} \mid^2 + \frac{1}{2}\frac{1}{\mu_0 c^2} \mid \vec{E} \mid^2 = \epsilon_0 \mid \vec{E} \mid^2 = \epsilon_0\left(E^2\cos^2(kz-\omega t) + 0^2\right) = \epsilon_0 E^2\cos^2(kz-\omega t),$$

and therefore the Poynting vector is

$$\vec{S}(\vec{r},t) = c\mathcal{E}\hat{k} = c\epsilon_0 E^2\cos^2(kz-\omega t)\,\hat{k},$$

where we can see that in case of linearly polarised plane wave the energy density and the Poynting vector do vary in time and space (direction of propagation, i.e. now z axis).

Aufgabe 2: Differentialoperatoren in Zylinderkoordinaten

(a) In Cartesian coordinate system the Nabla-operator is defined in the following way

$$\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} = \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix},$$

where the term $\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix}$ is actually a matrix. In order to derive the Nabla-operator in cylindrical coordinate system, one has to express the unit vectors and the partial derivatives with cylindrical coordinates. If we would like to change from Cartesian system (x, y, z) to cylindrical system (ρ, ϕ, z), the transformation rules for the coordinates are $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$.

First let us start with the partial derivatives. Here we can apply the chain rule so

$$\frac{\partial f}{\partial \rho} = \frac{\partial x}{\partial \rho}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \rho}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial \rho}\frac{\partial f}{\partial z},$$

$$\frac{\partial f}{\partial \phi} = \frac{\partial x}{\partial \phi}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial f}{\partial z},$$

$$\frac{\partial f}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial z}\frac{\partial f}{\partial z},$$

or in vectorial form

$$\begin{pmatrix} \frac{\partial f}{\partial \rho} \\ \frac{\partial f}{\partial \phi} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix},$$

which just defines the so-called Jacobian matrix and where f is an arbitrary differentiable function (which can be ignored). Using the transformation rules for the coordinates, we can deduct that $\frac{\partial x}{\partial \rho} = \cos\phi$, $\frac{\partial y}{\partial \rho} = \sin\phi$, $\frac{\partial x}{\partial \phi} = -\rho\sin\phi$, $\frac{\partial y}{\partial \phi} = \rho\cos\phi$, $\frac{\partial z}{\partial \rho} = \frac{\partial z}{\partial \phi} = \frac{\partial x}{\partial z} = \frac{\partial y}{\partial z} = 0$ and $\frac{\partial z}{\partial z} = 1$. Therefore

$$\partial\rho = \cos\phi\partial x + \sin\phi\partial y,$$

$$\partial\phi = -\rho\sin\phi\partial x + \rho\cos\phi\partial y,$$

$$\partial_z = \partial_z,$$

or in vectorial form (Jacobian matrix)

$$\begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}.$$

Since we do not perform transformation in the z component, there is no change in ∂_z. As regards ∂x and ∂y, we would like to express them with $\partial\rho$ and $\partial\phi$. We can realise from the Jacobian matrix that the first 2x2 sub-matrix is very similar to the rotation matrix (except for the ρ term, which is responsible for the Jacobian determinant). We can modify it in the following way

$$\begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{1}{\rho}\frac{\partial}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = R(\phi) \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix},$$

where $R(\phi)$ is the 2 dimensional rotation matrix with rotation of ϕ in the positive direction. And since we know that the rotation matrix has the following properties: $R(-\phi) = R^{-1}(\phi) = R^T(\phi)$, we can easily express ∂x and ∂y as

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = R(-\phi) \begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{1}{\rho}\frac{\partial}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{1}{\rho}\frac{\partial}{\partial \phi} \end{pmatrix}.$$

Now let us express the Cartesian unit vectors with cylindrical coordinates. To do this, we can introduce the cylindrical unit vectors as

$$\vec{e}_\rho = \frac{\frac{\partial \vec{x}}{\partial \rho}}{|\frac{\partial \vec{x}}{\partial \rho}|},$$

$$\vec{e}_\phi = \frac{\frac{\partial \vec{x}}{\partial \phi}}{|\frac{\partial \vec{x}}{\partial \phi}|},$$

$$\vec{e}_z = \frac{\frac{\partial \vec{x}}{\partial z}}{|\frac{\partial \vec{x}}{\partial z}|},$$

where $\vec{x} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$. After the calculation we obtain $\vec{e}_z = \vec{e}_z$ (since we do not perform transformation in the z component) and

$$\begin{pmatrix} \vec{e}_\rho \\ \vec{e}_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \end{pmatrix} = R(\phi) \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \end{pmatrix}.$$

We can express the Cartesian unit vectors with the same technique as above

$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \end{pmatrix} = R(-\phi) \begin{pmatrix} \vec{e}_\rho \\ \vec{e}_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \vec{e}_\rho \\ \vec{e}_\phi \end{pmatrix}.$$

Therefore the Nabla-operator can be expressed as

$$\vec{\nabla} = \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\phi\vec{e}_\rho - \sin\phi\vec{e}_\phi \\ \sin\phi\vec{e}_\rho + \cos\phi\vec{e}_\phi \\ \vec{e}_z \end{pmatrix} \begin{pmatrix} \cos\phi\partial_\rho - \frac{1}{\rho}\sin\phi\partial_\phi \\ \sin\phi\partial_\rho + \frac{1}{\rho}\cos\phi\partial_\phi \\ \partial_z \end{pmatrix}.$$

After the calculation we get 9 terms, but 4 of them cancel each other. The remaining 4 out of 5 terms contain sine and cosine squares such that we can combine them in pairs and finally we obtain the following 3 terms

$$\vec{\nabla} = \vec{e}_\rho \frac{\partial}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \vec{e}_z \frac{\partial}{\partial z}.$$

(b) Now we are about to determine the divergence operator $\vec{\nabla} \cdot \vec{A}$ for an arbitrary vector-field $\vec{A}$ in cylindrical coordinate system. To do this, we can use the result of question (a):

$$\vec{\nabla} \cdot \vec{A} = \left(\vec{e}_\rho \frac{\partial}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \vec{e}_z \frac{\partial}{\partial z}\right)\left(\vec{A}_\rho \vec{e}_\rho + \vec{A}_\phi \vec{e}_\phi + \vec{A}_z \vec{e}_z\right).$$

In order to calculate this product, we can consider the followings:

- $\vec{e_i}\vec{e_j} = \delta_{ij}$
- $\partial_\phi \vec{e}_\rho = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} = \vec{e}_\phi$ and $\partial_\phi \vec{e}_\phi = \begin{pmatrix} -\cos\phi \\ -\sin\phi \\ 0 \end{pmatrix} = -\vec{e}_\rho$ (here we used the equations from question (a))
- $\partial_\rho \vec{e}_\rho = \partial_\rho \vec{e}_\phi = \partial_\rho \vec{e}_z = \partial_\phi \vec{e}_z = \partial_z \vec{e}_\rho = \partial_z \vec{e}_\phi = \partial_z \vec{e}_z = 0.$

Therefore after the calculation we get the following non-zero terms

$$\vec{\nabla} \cdot \vec{A} = \vec{e}_\rho \frac{\partial}{\partial \rho} \left(A_\rho \vec{e_\rho} \right) + \vec{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} \left(A_\rho \vec{e_\rho} + A_\phi \vec{e_\phi} \right) + \vec{e}_z \frac{\partial}{\partial z} A_z \vec{e_z},$$

where we have to apply the product rule appropriately (e.g. $\vec{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} \left(A_\phi \vec{e_\phi} \right) = \vec{e}_\phi \frac{1}{\rho} \left[\frac{\partial}{\partial \phi} \left(A_\phi \right) \vec{e_\phi} + \frac{\partial}{\partial \phi} \left(\vec{e_\phi} \right) A_\phi \right] = \vec{e}_\phi \frac{1}{\rho} \left[\frac{\partial}{\partial \phi} \left(A_\phi \right) \vec{e_\phi} - \vec{e_\rho} A_\phi \right] = \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\phi$):

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial \rho} A_\rho + \frac{1}{\rho} A_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z = \left(\frac{\partial}{\partial \rho} + \frac{1}{\rho} \right) A_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z,$$

which can be rewritten as

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho A_\rho \right) + \frac{1}{\rho} \frac{\partial}{\partial \phi} \left(A_\phi \right) + \frac{\partial}{\partial z} \left(A_z \right),$$

since $\left(\frac{\partial}{\partial \rho} + \frac{1}{\rho} \right) A_\rho = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho A_\rho \right)$.