Exercises 18.05.2017: ANSWERS

• Problem 1

$$\hat{H} = \frac{1}{2m}\hat{\underline{p}}^2 + V(\hat{\underline{r}}) = -\frac{\hbar^2}{2m}\nabla^2 + V(\hat{\underline{r}})$$

We know that:
$$\begin{cases} [\hat{x}_i, G(\underline{p})] = i\hbar \dfrac{\partial G}{\partial \hat{p}_i} & \text{(a)} \\ [\hat{p}_i, F(\underline{x})] = -i\hbar \dfrac{\partial F}{\partial \hat{x}_i} & \text{(b)} \end{cases}$$

Moreover: $[A^2, B] = [A,B]A + A[A,B]$ (c)

Therefore

• $[\hat{H}, \hat{p}_i] = \left[\frac{1}{2m}\sum_j \hat{p}_j^2 + V(\hat{\underline{r}}), \hat{p}_i\right] = [V(\hat{\underline{r}}), \hat{p}_i]$

these commute

use (a):
$$[V(\hat{r}), \hat{p}_i] = i\hbar\frac{\partial V}{\partial x_i} \Rightarrow \boxed{[\hat{H}, \hat{\underline{p}}] = i\hbar\vec{\nabla}V}$$

• $[\hat{H}, \hat{r}_i] = \left[\frac{1}{2m}\sum_j \hat{p}_j^2 + V(\hat{\underline{r}}), \hat{r}_i\right]$

commute

$$= \frac{1}{2m}\sum_j [\hat{p}_j^2, \hat{r}_i] = \frac{1}{2m}\sum_j \Big\{ \underbrace{[\hat{p}_j, \hat{r}_i]}_{=-i\hbar\delta_{ij}}\hat{p}_j + \hat{p}_j\underbrace{[\hat{p}_j, \hat{r}_i]}_{=-i\hbar\delta_{ij}} \Big\}$$

$$= \frac{1}{2m}\sum_j (-2i\hbar)\delta_{ij}\hat{p}_j$$

$$= \frac{-i\hbar}{m}\hat{p}_i \Rightarrow \boxed{[\hat{H}, \hat{\underline{r}}] = -\frac{i\hbar}{m}\hat{\underline{p}}}$$

E2)

- Problem 2

The linear Transformation

$$\varphi_\beta = \sum_{\alpha=1}^{D} a_{\beta\alpha} \psi_\alpha \qquad \beta = 1, 2, \dots, D$$

contains D^2 coefficients $a_{\beta\alpha}$. How many constraints of the form

$$(\varphi_\alpha, \varphi_\beta) \equiv \int \varphi_\alpha^*(q)\, \varphi_\beta(q)\, dq = \delta_{\alpha\beta}$$

we have? Let us count them:

$(\varphi_1, \varphi_1), (\varphi_1, \varphi_2), \dots, (\varphi_1, \varphi_D)$ ← D constraints

$(\varphi_2, \varphi_2), (\varphi_2, \varphi_3), \dots, (\varphi_2, \varphi_D)$ ← $(D-1)$ constraints

$\vdots$

$(\varphi_{D-1}, \varphi_{D-1}), (\varphi_{D-1}, \varphi_D)$ ← 2 constraints

(φ_D, φ_D) ← 1 constraint

Therefore, the total number C of constraints is:

$$C = \underbrace{D}_{=D-0} + (D-1) + (D-2) + \dots + 2 + \underbrace{1}_{=D-(D-1)}$$

$$= \sum_{k=0}^{D-1} (D-k) = \sum_{k=0}^{D-1} D - \sum_{k=0}^{D-1} k$$

$$= D^2 - \frac{D(D-1)}{2}$$

$$= \frac{D(D+1)}{2} < D^2 \Rightarrow \text{infinitely many ways!}$$

• Problem 3

$$\psi(\underline{r},t) = A e^{i\chi + i\omega t} + B e^{-i\chi + i\omega t} \equiv A e^{i a_+} + B e^{-i a_-}$$

$$a_\pm \equiv \chi \pm \omega t$$

$$\omega \equiv \frac{E}{\hbar} \; ; \; \chi \equiv \frac{\vec{r}.\vec{p}}{\hbar}$$

$$\rho = |\psi|^2 = (A^* e^{-i\chi - i\omega t} + B^* e^{i\chi - i\omega t})(A e^{i\chi + i\omega t} + B e^{-i\chi + i\omega t})$$

$$= |A|^2 + A^* B e^{-2i\chi\omega} + B^* A e^{2i\chi\omega} + |B|^2$$

$$= |A|^2 + |B|^2 + 2|AB| \cos(\alpha - \beta + 2\chi)$$

where
$$A = |A| e^{i\alpha} \; ; \; B = |B| e^{i\beta}$$

Therefore $\frac{\partial \rho}{\partial t} = 0$

$$\vec{J} = \frac{\hbar}{2mi}\left[\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right]$$

Since
$$\frac{\partial}{\partial x}\psi = i\frac{\partial \chi}{\partial x}\left(A e^{i(\chi + \omega t)} - B e^{-i(\chi - \omega t)}\right)$$

and
$$\frac{\partial \chi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\vec{r}.\vec{p}}{\hbar}\right) = \frac{P_x}{\hbar} \Rightarrow$$

$$\Rightarrow \vec{\nabla}\psi = \frac{i\vec{\nabla}\vec{p}}{\hbar}\left[A e^{i\underbrace{(\chi + \omega t)}_{\equiv a_+}} - B e^{-i\underbrace{(\chi - \omega t)}_{\equiv a_-}}\right]$$

$$a_+ + a_- = 2\chi$$

Therefore

$$\frac{2mi}{\hbar}\vec{J} = \Big\{ (A^* e^{-i\theta_+} + B^* e^{+i\theta_-})(A e^{i\theta_+} - B e^{-i\theta_-}) +$$

because $\vec{\nabla}\psi^* \propto -i$

$$+ (A e^{i\theta_+} + B e^{-i\theta_-})(A^* e^{-i\theta_+} - B^* e^{i\theta_-}) \Big\} \frac{\vec{\nabla P} i}{\hbar}$$

$$\Leftrightarrow \vec{J} = \frac{\vec{\nabla P}}{2m}\Big[|A|^2 - \cancel{A^* B e^{-2ix}} + \cancel{B^* A e^{2ix}} - |B|^2 +$$

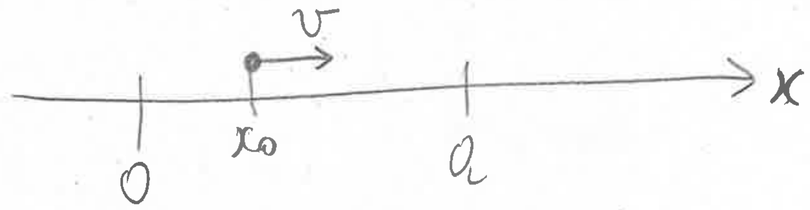
$$+ |A|^2 - \cancel{AB^* e^{2ix}} + \cancel{BA^* e^{-2ix}} - |B|^2 \Big]$$

$$\boxed{\vec{J} = \frac{1}{m}\vec{\nabla P}\left(|A|^2 - |B|^2\right)}$$

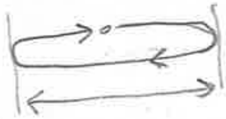
If $|A| = |B|$ the two counterpropagating waves form a STATIONARY wave.

Since $\vec{J}$ is independent of $x \Rightarrow \vec{\nabla}\cdot\vec{J} = 0$ c.v.d

• Problem 4



Round Trip time $t_{RT} = \frac{2a}{v}$

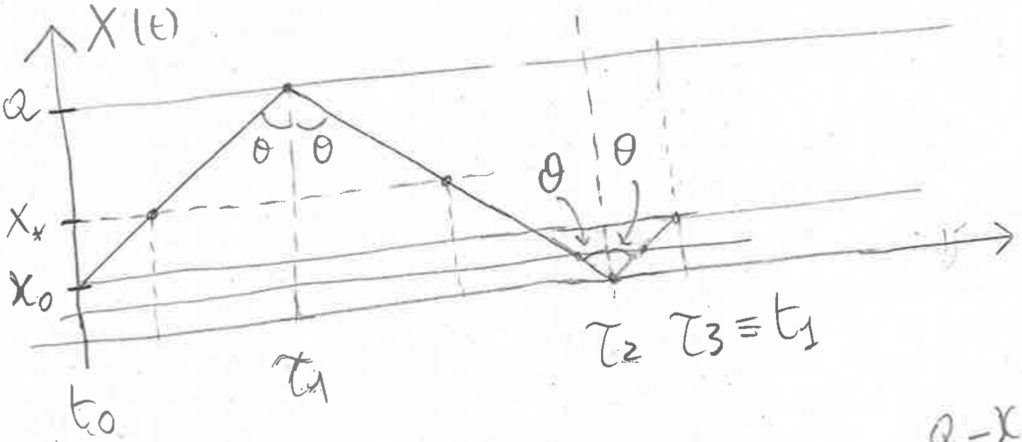


let $x = X(t)$ the position of the particle at time t.

Suppose that at $t = t_0$ the particle is in $x_0 > 0$ and moves from left to right:

$$x_0 = X(t_0)$$

The plot of the Trajectory is:



Step 1: move from x_0 to $x = a$ in a time $\frac{a - x_0}{v} = t_1 - t_0$

Step 2: move backwards to $x = 0$ in a time $t_2 - t_1 = \frac{a}{v}$

Step 3: move forward from $x = 0$ to $x = x_0$ in a time $t_3 - t_2 = \frac{x_0}{v}$

Then $t_3 - t_0 = (t_3 - t_2) + (t_2 - t_1) + (t_1 - t_0)$

$$= \frac{\cancel{x_0}}{v} + \frac{a}{v} + \frac{a - \cancel{x_0}}{v} = \frac{2a}{v} = t_{RT}$$

By hypothesis, t_* is randomly picked (with uniform distribution) between t_0 and t_1.
This means that the PROBABILITY DENSITY FUNCTION (pdf) $f_t(t)$ is:

$$(1.6) \qquad f_t(t) = \begin{cases} \dfrac{1}{t_1 - t_0}, & t_0 \le t \le t_1 \\ 0, & \text{otherwise} \end{cases}$$

So, if t is a random variable distributed according to (1.6), what is the distribution of the random variable x defined by

$$x = X(t) \quad ?$$

This is a special case of the following general problem:
« Given the independent random variable x with pdf $f_x(x)$, and given the dependent random variable y defined by

$$y = g(x),$$

find the pdf $f_y(y)$ of the random variable y. »

The answer is simply:

$$f_y(y) = \int \delta(g(x) - y) f_x(x)\, dx$$

From the formula for a delta of function, we know that:

$$\delta(g(x)-y) = \sum_{i=1}^{n} \frac{\delta(x-x_i)}{|g'(x_i)|}$$

where x_i are the points where $g(x_i)-y=0$

This equation sets n functions, formally given by:

$$x_i = g^{-1}(y) \equiv \xi_i(y)$$

Therefore

$$f_y(y) = \sum_{i=1}^{n} \frac{1}{|g'(\xi_i(y))|} f_x(\xi_i(y))$$

* example *

$$g(x) = ax+b$$
$$g'(x) = a$$
$$g(x)-y=0 \Leftrightarrow ax+b-y=0 \Rightarrow x = \frac{y-b}{a}$$

$$f_y(y) = \int \delta(ax+b-y) f_x(x)\,dx$$

$$= \frac{1}{|a|}\int \delta\left(x+\frac{b-y}{a}\right) f_x(x)\,dx$$

$$= \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right) \quad \checkmark$$

In our case, let $f_x(x)$ the unknown fdd.

Then:

$$f_x(x) = \sum_{i=1}^{n} \frac{1}{|\dot{X}(t_i)|} f_t(t_i)$$

In our case, from the figure, we have:

$$X(t) = \begin{cases} x_0 + vt, & t_0 < t < \tau_1 \\ a - vt & \tau_1 < t < \tau_2 \\ vt & \tau < t < t_1 \end{cases}$$

and

$$\dot{X}(t) = \begin{cases} v & t_0 < \tau < \tau_1 \\ -v & \tau_1 < t < \tau_2 \\ v & \tau < t < t_1 \end{cases} \Rightarrow |\dot{x}(t)| = v$$

There are always 2 solutions of the equation

$$X(t) = x$$

If $x > x_0$ then $\begin{cases} x_0 + vt = x & \Rightarrow t = (x - x_0)/v \\ a - vt = x & \Rightarrow t = (a - x)/v \end{cases}$

If $x \leq x_0$ then $\begin{cases} a - vt = x & \Rightarrow t = (a - x)/v \\ vt = x & \Rightarrow t = \frac{x}{v} \end{cases}$

However, $f_t(t) = \frac{1}{t_{RT}} = \frac{v}{2a}$ indep on t, therefore

$$f_x(x) = \frac{1}{\cancel{v}}\frac{\cancel{v}}{2a} + \frac{1}{\cancel{v}}\frac{\cancel{v}}{2a} = \frac{1}{a}$$

So, the classical probability is:

$$P_{cl}(x)\,dx = \frac{dx}{a}$$ ← Still uniform

- Problem 5

$$T = \frac{1}{2}mv^2 = \frac{1}{2}\,0.1\times10^{-3}\,\mathrm{Kg}\;1\,\frac{m^2}{s^2}$$

$$= 5\times10^{-5}\,\mathrm{J}$$

$$[E_n] = \frac{[\hbar^2]}{[m][a^2]} = \frac{E^2t^2}{mL^2}$$

$$= \frac{E^2}{mv^2} = E$$

$$E_n = n^2\frac{\hbar^2\pi^2}{2ma^2}$$

$$= n^2\,5.5\times10^{-58}\,\mathrm{J}$$

$$T - \nu^2E_n = 0 \Rightarrow \nu = \sqrt{\frac{T}{E_n}} \simeq 3.02\times10^{26}$$

The answer is:

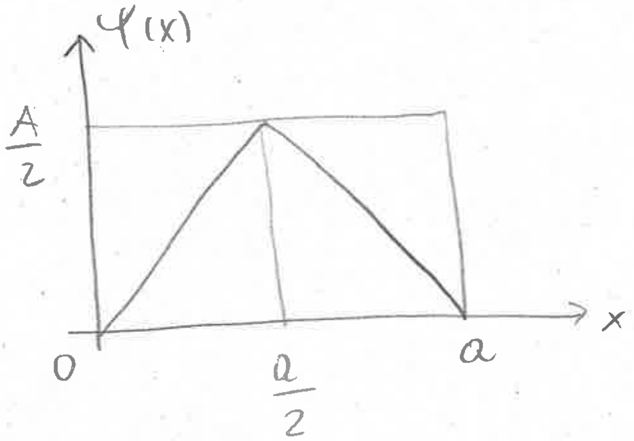
$$n = 301\,841\,231\,512\,224\,569\,876\,245\,095$$

$$h = 6.626\times10^{-34}\,\mathrm{J\cdot S}$$

$$\hbar = 1.05457\times10^{-34}\,\mathrm{J\cdot S}$$

- **Problem 6**

$$\psi(x) = \begin{cases} A\,\frac{x}{a} & , \quad 0 < x < \frac{a}{2} \\ A\left(1 - \frac{x}{a}\right) & , \quad \frac{a}{2} < x < a \end{cases}$$



– Normalization –

Assume $A > 0$ (a phase factor in front of a wave function does not affect any result!)

$$\int_0^a |\psi(x)|^2 dx = \int_0^{a/2} A^2 \frac{x^2}{a^2} dx + \int_{a/2}^{a} A^2 \left(1 - \frac{x}{a}\right)^2 dx$$

$$= A^2 \left\{ \frac{a}{24} + \frac{a}{24} \right\} = A^2 \frac{a}{12} \Rightarrow A = 2\sqrt{\frac{3}{a}}$$

– Expansion –

The eigenfunctions of $\hat{H}$ are:

$$u_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

Therefore, if $\psi(x) = \sum_{n=1}^{\infty} A_n u_n(x)$

then

$$A_n = \int_0^a u_n^*(x)\,\psi(x)\,dx$$

$$= \sqrt{\frac{2}{a}}\sqrt{\frac{12}{a}}\left[\int_0^{a/2} \frac{x}{a}\sin\left(\frac{n\pi x}{a}\right)dx + \right.$$

$$\left. + \int_{a/2}^{a}\left(1-\frac{x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dx\right]$$

In the first integral we change variable $u = \frac{\pi x}{a} \Rightarrow dx = \frac{a}{\pi}du$

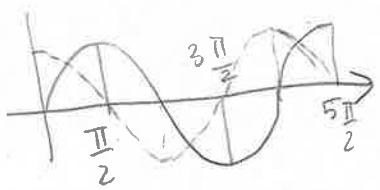
and in the second one $v = \pi\left(1-\frac{x}{a}\right) \Rightarrow dx = -\frac{a}{\pi}dv$

Therefore

$$A_n = \frac{\sqrt{24}}{\cancel{a}}\left[\frac{\cancel{a}}{\pi^2}\int_0^{\pi/2} u\sin(nu)\,du + \left(-\frac{a}{\pi}\right)\int_{\pi/2}^{0}\frac{v}{\pi}\underbrace{\sin[n(\pi-v)]}_{=(-1)^{n+1}\sin(nv)}dv\right]$$

$$= \frac{\sqrt{24}}{\pi^2}\int_0^{\pi/2}\alpha\left[1+(-1)^{n+1}\right]\sin(n\alpha)\,d\alpha$$

$$= \frac{\sqrt{24}}{\pi^2}\underbrace{\left[1-(-1)^n\right]}_{\substack{=0 \text{ if } n \text{ EVEN} \\ =2 \text{ if } n \text{ ODD}}}\underbrace{\int_0^{\pi/2}\alpha\sin(n\alpha)\,d\alpha}_{=\frac{1}{n^2}\left\{\sin\left(\frac{n\pi}{2}\right)-\underbrace{\frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)}_{=0}\right\}\ n \text{ odd}}$$



$$\Rightarrow \sin\left(\frac{n\pi}{2}\right)\Big|_{n\,odd} = \begin{cases} 1 & n=1 \\ -1 & n=3 \\ 1 & n=5 \\ \vdots & \end{cases}$$

$$A_n = \frac{\sqrt{24}}{\pi^2} 2 \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \checkmark$$

Therefore:

$$\text{Prob}(E_n) = |A_n|^2 = \begin{cases} \frac{96}{\pi^4 n^4} & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

It is easy to check that:

$$\sum_{n=1}^{\infty} |A_n|^2 = 1$$

$$\sum_{n=1}^{\infty} |A_n|^2 = \sum_{n \in odd} \frac{96}{\pi^4 n^4} = \frac{96}{\pi^4} \sum_{n \in odd} \frac{1}{n^4}$$

but $\sum_n \frac{1}{n^4} = \sum_{n \in odd} \frac{1}{n^4} + \sum_{n \in even} \frac{1}{n^4} \Rightarrow$

$$\Rightarrow \sum_{n \in odd} \frac{1}{n^4} = \sum_n \frac{1}{n^4} - \underbrace{\sum_{n \in even} \frac{1}{n^4}}$$

$$= \sum_{k=1}^{\infty} \frac{1}{(2k)^4} = \frac{1}{16} \sum_n \frac{1}{n^4}$$

$$= \left(1 - \frac{1}{16}\right) \underbrace{\sum_n \frac{1}{n^4}}_{= \pi^4/90} \Rightarrow$$

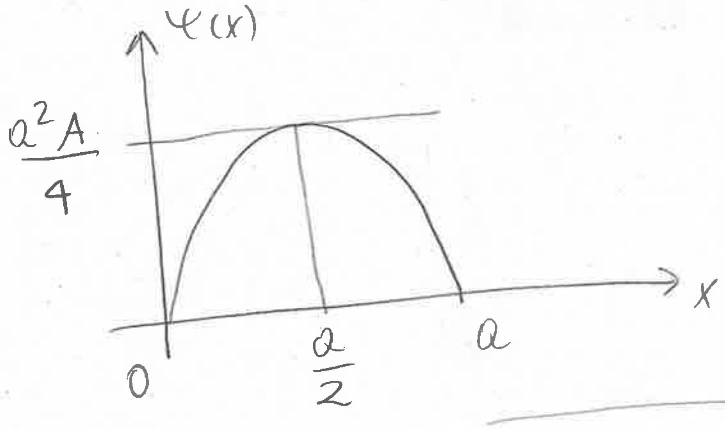
$$\Rightarrow \sum_{n \in odd} \frac{1}{n^4} = \frac{15}{16} \frac{\pi^4}{90}$$

Therefore $\sum_{n=1}^{\infty} |A_n|^2 = \frac{96}{\pi^4} \frac{15}{16} \frac{\pi^4}{90} = 1$

because $96 \times 15 = 90 \times 16 = 1440$

- Problem 7

$$\psi(x) = A\,x(a-x)$$



- $\int_0^a |\psi(x)|^2 dx = \frac{a^5}{30} A^2 \Rightarrow \boxed{\psi(x) = \sqrt{\frac{30}{a^5}}\, x(a-x)}$

- $\langle \hat{x} \rangle = \int_0^a x |\psi(x)|^2 dx = \frac{a}{2}$

- $\langle \hat{p} \rangle = -i\hbar \int_0^a \underset{\text{even}}{\psi^*(x)}\, \underset{\text{x odd}}{\psi'(x)}\, dx = 0$

- $\langle \hat{H} \rangle = \frac{-\hbar^2}{2m} \int_0^a \psi^*(x) \underbrace{\psi''(x)}_{= -2\sqrt{\frac{30}{a^5}}} dx = \frac{10\hbar^2}{2ma^2} = \frac{5\hbar^2}{ma^2}$

$$A_n = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \sqrt{\frac{30}{a^5}}\, x(a-x)\, dx$$

$$A_n = \frac{8\sqrt{15}}{\pi^3 n^3} \underbrace{\frac{1-(-1)^n}{2}}_{\substack{=1 \quad n\,\text{odd} \\ 0 \quad n\,\text{even}}}$$

Therefore $$A_n = \frac{8\sqrt{15}}{\pi^3} \frac{1}{n^3} \times \begin{cases} 1 & n\,\text{odd} \\ 0 & n\,\text{even} \end{cases}$$

Therefore $$|A_n|^2 \frac{\pi^3}{8\sqrt{15}} = \frac{1}{n^3} \qquad n\,\text{odd}$$

and: $$\frac{|A_3|^2}{|A_1|^2} = \left(\frac{1}{3^3}\right)^2 = \frac{1}{729}$$

• Problem 8

$$\frac{d^2u}{dx^2} + k^2 u = 0$$

If $u(x) = Ae^{ikx} + Be^{-ikx}$

Then $u'(x) = ik\left(Ae^{ikx} - Be^{-ikx}\right)$

$$u''(x) = (ik)^2 u(x) = -k^2 u(x)$$

The boundary conditions are:

$$u(0) = 0 \quad \text{and} \quad u(a) = 0$$

—LONG PROCEDURE—

Let $v = v(x)$ another solution of

$$v'' + k^2 v = 0$$

with BC: $\quad v(0) = 0 = v(a)$

If v is independent from u, the Wronskian $W(x)$ must be non-zero:

$$W = \begin{vmatrix} u(x) & v(x) \\ u'(x) & v'(x) \end{vmatrix} = \begin{vmatrix} Ae^{ikx} + Be^{-ikx} & v(x) \\ ik\left(Ae^{ikx} - Be^{-ikx}\right) & v'(x) \end{vmatrix}$$

$$= \left(\underset{\sim}{A}e^{ikx} + \underset{0}{B}e^{-ikx}\right)v'(x) - ik\left(\underset{\sim}{A}e^{ikx} - \underset{0}{B}e^{-ikx}\right)v(x)$$

$$W = Ae^{ikx}(v' - ikv) + Be^{-ikx}(v' + ikv)$$

let us take the derivative:

$$\frac{dW}{dx} = ikAe^{ikx}(v' - ikv) + Ae^{ikx}(v'' - ikv') +$$

$$- ikBe^{-ikx}(v' + ikv) + Be^{-ikx}(v'' + ikv')$$

$$= Ae^{ikx}(\cancel{ikv'} + \underbrace{k^2v + v''}_{=0 \text{ because by hypothesis } v \text{ is a solution}} - \cancel{ikv'})$$

$$+ Be^{-ikx}(\cancel{-ikv'} + \underbrace{k^2v^2 + v''}_{=0} + \cancel{ikv'})$$

$$= 0$$

But $\frac{dW}{dx} = 0 \Leftrightarrow W(x) = \text{const.}$

Therefore I can evaluate $W(x)$ at $x = 0$:

$$W(0) = \underbrace{u(0)}_{=0} v'(0) - \underbrace{v(0)}_{=0} u'(0) = 0$$

because of boundary conditions

Therefore u and v are NOT independent

– SHORT PROCEDURE –

$$W = uv' - vu' \Rightarrow W' = \cancel{u'v'} + uv'' - \cancel{v'u'} - vu''$$
$$= u(-k^2v) - v(-k^2u) = 0 \Rightarrow W = \text{const.}$$

$W(0) = 0$ <u>end</u>