— Homeworks 22.06.2017 — A. AIELLO

Problem 1. Given the polarization basis $\{|x\rangle, |y\rangle\}$, calculate:

a) The state vectors of a photon linearly polarized at $\pm 45^\circ$ with respect to the x-axis. Show that the matrix connecting these states, denoted $|\pm 45^\circ\rangle$, with $|x\rangle$ and $|y\rangle$, is unitary.

b) The matrix, in the basis $\{|x\rangle, |y\rangle\}$, of a polaroid filter that transmits y-polarized light. Show that this matrix is a projector

c) Same as point b), but now the polarizer transmits light polarized at x':

$$\vec{e}_{x'} = \cos\theta\, \vec{e}_x + \sin\theta\, \vec{e}_y$$

d) Same as point b) but the polarizer transmits R-polarized (L-polarized) light

e) A "λ/n" plate is a transparent anysotropic plate that introduces a phase difference between the x and y components of the electric field, given by:

$$\Delta\phi = \frac{2\pi}{n} : (E_x, E_y) \to \left(E_x e^{i\phi_x}, E_y e^{i\phi_y}\right) ; \Delta\phi = \phi_y - \phi_x$$

Write, in the $\{|x\rangle, |y\rangle\}$ basis, the matrices representing a $\lambda/4$ and a $\lambda/2$ plates.

a) An electric field polarized at $\pm 45°$ has components with respect to the x,y axes:

$$+45°: \quad E_x = E_y \quad ; \quad -45°: \quad E_x = -E_y$$

Since, by definition,

$$|\psi\rangle = \psi_x |x\rangle + \psi_y |y\rangle$$

where $\quad \psi_x = \frac{E_x}{\sqrt{|E_x|^2 + |E_y|^2}} \quad ; \quad \psi_y = \frac{E_y}{\sqrt{|E_x|^2 + |E_y|^2}}$

we conclude that, up to an irrelevant multiplicative phase,

$$|\pm 45°\rangle = \frac{1}{\sqrt{2}} \left(\pm |x\rangle + |y\rangle \right)$$

Let $\hat{U}$ be an operator such that:

$$|+45°\rangle = \hat{U}|x\rangle \quad \Rightarrow$$
$$|-45°\rangle = \hat{U}|y\rangle$$

$$\Rightarrow \hat{U} \doteq U = \begin{bmatrix} \langle x|\hat{U}|x\rangle & \langle x|\hat{U}|y\rangle \\ \langle y|\hat{U}|x\rangle & \langle y|\hat{U}|y\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \langle x|+45°\rangle & \langle x|-45°\rangle \\ \langle y|+45\rangle & \langle y|-45°\rangle \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

So, if $|x\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|y\rangle \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then

$$|+45^\circ\rangle \doteq U \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|-45^\circ\rangle \doteq U \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The matrix U is isometric:

$$U^+U = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and unitary:

$$U^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = U^+$$

Note: An operator $\hat{U}$ is isometric if, given

$$|\phi\rangle = \hat{U}|\psi\rangle \overset{CD}{\Longleftrightarrow} \langle\psi|\hat{U}^+ = \langle\phi|$$

then $\langle\phi|\phi\rangle = \langle\psi|\hat{U}^+\hat{U}|\psi\rangle = \langle\psi|\psi\rangle \Rightarrow \hat{U}^+\hat{U} = \hat{I}$

If an isometric operator has an inverse, then it is unitary:

$\hat{U}^+\hat{U} = \hat{I}$ if $\hat{U}^{-1}$ exists, then $\hat{U}^+\hat{U} = \hat{I} \Leftrightarrow (\hat{U}^+\underbrace{\hat{U})\hat{U}^{-1}}_{=\hat{I}} = \hat{U}^{-1}$

$\Leftrightarrow \hat{U}^+ = \hat{U}^{-1}$

b) Let $\hat{P}_y$ denote the operator transmitting y-polarized light for any state $|\psi\rangle$. E4)

We know that y-polarization passes across it but x-polarization is completely absorbed. Therefore:

$$\begin{cases} \hat{P}_y |y\rangle = |y\rangle \\ \hat{P}_y |x\rangle = 0 \end{cases}$$

This implies:

$$\hat{P}_y \doteq P_y = \begin{bmatrix} \langle x|\hat{P}_y|x\rangle & \langle x|\hat{P}_y|y\rangle \\ \langle y|\hat{P}_y|x\rangle & \langle y|\hat{P}_y|y\rangle \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

A matrix M is a projector when $M^2 = M$. In this case

$$P_y^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = P_y \quad \underline{\text{c.v.d.}}$$

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c) Let $\hat{P}_{x'}$ the operator representing the polarizer, where

$$|x'\rangle = \cos\theta\,|x\rangle + \sin\theta\,|y\rangle$$

A state $|y'\rangle$ orthogonal to $|x'\rangle$ is clearly:

$$|y'\rangle = -\sin\theta\,|x\rangle + \cos\theta\,|y\rangle$$

Then, by definition:

$$\hat{P}_{x'}|x'\rangle = |x'\rangle$$

$$\hat{P}_{x'}|y'\rangle = 0$$

Therefore:

$$\hat{P}_{x'} \doteq P_{x'} = \begin{bmatrix} \langle x|\hat{P}_{x'}|x\rangle & \langle x|\hat{P}_{x'}|y\rangle \\ \langle y|\hat{P}_{x'}|x\rangle & \langle y|\hat{P}_{y'}|y\rangle \end{bmatrix} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$\langle x|\hat{P}_{x'}|x'\rangle = \langle x|\hat{P}_{x'}|x\rangle\overbrace{\langle x|x'\rangle}^{\cos\theta} + \langle x|\hat{P}_{x'}|y\rangle\overbrace{\langle y|x'\rangle}^{\sin\theta} = \overbrace{\langle x|x'\rangle}^{\cos\theta}$$

$$\langle y|\hat{P}_{x'}|x'\rangle = \langle y|\hat{P}_{x'}|x\rangle\langle x|x'\rangle + \langle y|\hat{P}_{x'}|y\rangle\langle y|x'\rangle = \overbrace{\langle y|x'\rangle}^{\sin\theta}$$

$$\langle x|\hat{P}_{x'}|y'\rangle = \langle x|\hat{P}_{x'}|x\rangle\overbrace{\langle x|y'\rangle}^{-\sin\theta} + \langle x|\hat{P}_{x'}|y\rangle\overbrace{\langle y|y'\rangle}^{\cos\theta} = 0$$

$$\langle y|\hat{P}_{x'}|y'\rangle = \langle y|\hat{P}_{x'}|x\rangle\langle x|y'\rangle + \langle y|\hat{P}_{x'}|y\rangle\langle y|y'\rangle = 0$$

These equations can be rewritten as:

$$\begin{aligned} a\cos\theta + b\sin\theta &= \cos\theta \\ c\cos\theta + d\sin\theta &= \sin\theta \\ -a\sin\theta + b\cos\theta &= 0 \\ -c\sin\theta + d\cos\theta &= 0 \end{aligned} \iff \left[\begin{array}{cc|cc} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ \hline 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{array}\right] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \cos\theta \\ 0 \\ \sin\theta \\ 0 \end{bmatrix}$$

This matrix equation has the form: (block-diagonal)

$$\begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} \underline{u} \\ \underline{v} \end{bmatrix} = \begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \end{bmatrix}$$

with $\underline{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\underline{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, $\underline{\alpha} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$; $\underline{\beta} = \begin{bmatrix} \sin\theta \\ 0 \end{bmatrix}$

$$D(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \Rightarrow D^{-1}(\theta) = D(-\theta)$$

Therefore:

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta\sin\theta \\ \sin^2\theta \end{bmatrix}$$

Finally:

$$P_{x'} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

Clearly:

$$P_{x'} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos^3\theta + \sin^2\theta\cos\theta \\ \sin^2\theta\cos\theta + \sin^3\theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \quad \underline{\text{c.v.d}}$$

Moreover:

$$P^2_{x'} = \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix} \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix} = \begin{bmatrix} c^4 + s^2c^2 & sc^3 + cs^3 \\ sc^3 + cs^3 & s^2c^2 + s^4 \end{bmatrix} = \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix} = P_{x'}$$

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d) Let $\hat{P}_R$ and $\hat{P}_L$ the two sought operators, where:

$$\begin{cases} \hat{P}_R |R\rangle = |R\rangle \\ \hat{P}_R |L\rangle = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \hat{P}_L |R\rangle = 0 \\ \hat{P}_L |L\rangle = |L\rangle \end{cases}$$

From these equations it follows at once:

$$\hat{P}_R = |R\rangle\langle R| \quad \text{and} \quad \hat{P}_L = |L\rangle\langle L|$$

We could, obviously, have done the same in the previous point c), but it is more instructive to see both the "pedantic" method in c) and the "smart" method in d)

So

$$\hat{P}_R \doteq P_R = \begin{bmatrix} \langle x|R\rangle\langle R|x\rangle & \langle x|R\rangle\langle R|y\rangle \\ \langle y|R\rangle\langle R|x\rangle & \langle y|R\rangle\langle R|y\rangle \end{bmatrix}$$

where

$$|R\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle + i|y\rangle\right) \Rightarrow$$

$$|L\rangle = \frac{1}{\sqrt{2}}\left(|x\rangle - i|y\rangle\right)$$

$$\Rightarrow P_R = \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}, \quad P_R^2 = \frac{1}{4}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 2 & -2i \\ 2i & 2 \end{bmatrix} = P_R$$

and

$$\hat{P}_L \doteq P_L = \begin{bmatrix} \langle x|L\rangle\langle L|x\rangle & \langle x|L\rangle\langle L|y\rangle \\ \langle y|L\rangle\langle L|x\rangle & \langle y|L\rangle\langle L|y\rangle \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = P_R^* = P_R^T \Rightarrow P_L^2 = (P_R^*)^2 = (P_R^2)^*$$

$$= P_R^* = P_L$$

and

$$P_R P_L = \frac{1}{4}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1-1 & i-i \\ i-i & -1+1 \end{bmatrix} = 0$$

$$P_L P_R = P_R^* P_L^* = (P_R P_L)^* = 0$$

e) Since $(E_x, E_y) \to e^{i\phi_x}(E_x, E_y e^{i\Delta\phi}) \Rightarrow$

$$\Rightarrow |\psi\rangle = \psi_x|x\rangle + \psi_y|y\rangle \to \psi_x|x\rangle + \psi_y e^{i\Delta\phi}|y\rangle$$

Let $\hat{\Lambda}$ be the operator performing this:

$$\hat{\Lambda}|\psi\rangle = \psi_x|x\rangle + \psi_y e^{i\Delta\phi}|y\rangle = \psi_x(\hat{\Lambda}|x\rangle) + \psi_y(\hat{\Lambda}|y\rangle)$$

Therefore $\begin{cases} \hat{\Lambda}|x\rangle = |x\rangle \\ \hat{\Lambda}|y\rangle = e^{i\Delta\phi}|y\rangle \end{cases} \Rightarrow$

$$\Rightarrow \hat{\Lambda} \doteq \Lambda = \begin{bmatrix} \langle x|\hat{\Lambda}|x\rangle & \langle x|\hat{\Lambda}|y\rangle \\ \langle y|\hat{\Lambda}|x\rangle & \langle y|\hat{\Lambda}|y\rangle \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\Delta\phi} \end{bmatrix}$$

$\frac{\lambda}{4}$ plate $\Leftrightarrow \Delta\phi = \frac{2\pi}{4} = \frac{\pi}{2} \Rightarrow e^{i\Delta\phi} = e^{i\pi/2} = i$

$\frac{\lambda}{2}$ plate $\Leftrightarrow \Delta\phi = \frac{2\pi}{2} = \pi \Rightarrow e^{i\Delta\phi} = e^{i\pi} = -1$

Therefore:

$$\Lambda(\lambda/4) = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

$$\Lambda(\lambda/2) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that:

$$\hat{\Lambda}(\lambda/4)|\pm 45°\rangle \doteq \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} \pm 1 \\ 1 \end{bmatrix} = \begin{cases} |R\rangle, & + \\ -|L\rangle, & - \end{cases}$$

So, a $\lambda/4$ plate is used to produce circularly polarized light.