1. Introduction

1.1 The Schrödinger equation

(brief reminder)

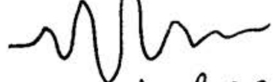
heat radiation
$\rightarrow$ Planck spectrum $\sim \dfrac{\omega^3}{e^{\hbar\omega/kT}-1}$

Planck (1900), Einstein (1905): energy quantized for light

$$\boxed{E = \hbar\omega}$$

$$\hbar = \frac{h}{2\pi} \approx 10^{-34}\,Js$$

h: "Planck's quantum"



packets of energy?

~~de Broglie~~

Einstein: photon momentum

$$\boxed{p = \hbar k}$$

$$k = \frac{2\pi}{\lambda}$$

(note: goes together with $E=\hbar\omega$ due to relativity)

de Broglie (1924): paticles $\rightarrow$ matter waves?
also with $E=\hbar\omega$, $p=\hbar k$

but now
$$E = \frac{p^2}{2m}$$

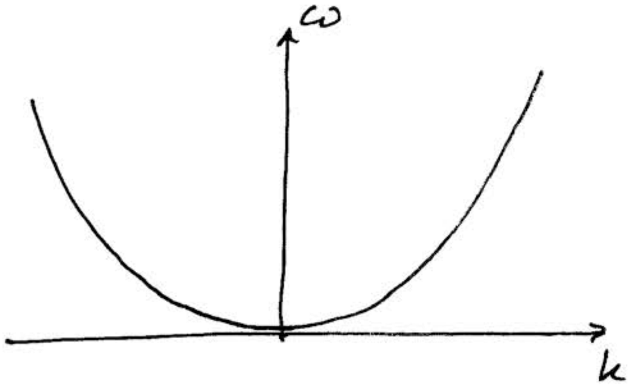
instead of $E=pc$

[actually, de Broglie used the relativistic energy
$$E = \sqrt{m^2c^4 + p^2c^2}$$]

Goal: derive <u>linear</u> wave equation
that yields correct dispersion relation $\omega = \omega(k)$

$$E = \frac{p^2}{2m}$$

$$\hbar\omega = \frac{\hbar^2 k^2}{2m}$$



In free space: no x or t special $\Rightarrow$
solutions will be plane waves

$$\psi_{(x,t)} \sim e^{i(kx - \omega t)}$$

$$i\hbar\partial_t \psi = \overbrace{\hbar\omega}^{E} \psi$$

$$-i\hbar\partial_x \psi = \underbrace{\hbar k}_{p} \psi$$

$\Rightarrow$ use:

$$\boxed{i\hbar\partial_t \psi = \frac{(-i\hbar\partial_x)^2}{2m}\psi + V(x)\psi}$$

Schrödinger equation
1926

compatible with
$E = \frac{p^2}{2m} + V(x)$
(& keeps conservation of $|\psi|^2$, see below)

alternative & equivalent:
Heisenberg's "matrix mechanics"
1925

"Standing waves" yield discrete energy levels

Ansatz $\quad \psi_n(x,t) = \phi_n(x)\, e^{-\frac{i}{\hbar}E_n t}$

with $\quad \hat{H}\phi_n \overset{!}{=} E_n \phi_n$

energy eigenvalues

and with the "Hamiltonian operator"

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$$

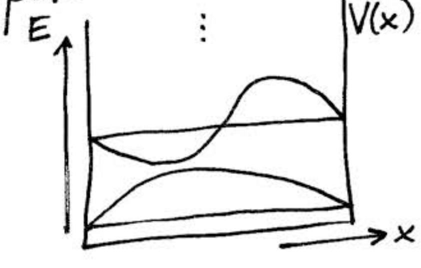
$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \quad \text{momentum operator}$$

Remember:
E_n are real and ϕ_n can be chosen as orthonormal basis

$$\underbrace{\underbrace{\langle\phi_n|}_{\text{"bra"}}\underbrace{|\phi_m\rangle}_{\text{"ket"}}}_{\text{scalar product between vectors of Hilbert space}} = \int \phi_n^*(x)\phi_m(x)\,dx = \delta_{n,m}$$

Examples:

particle in a box

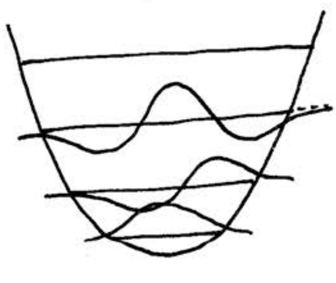


$$E_n = \frac{\hbar^2 k_n^2}{2m}$$

$$k_n = n\frac{\pi}{L}$$

$n = 1, 2, 3, \ldots$

harmonic oscillator

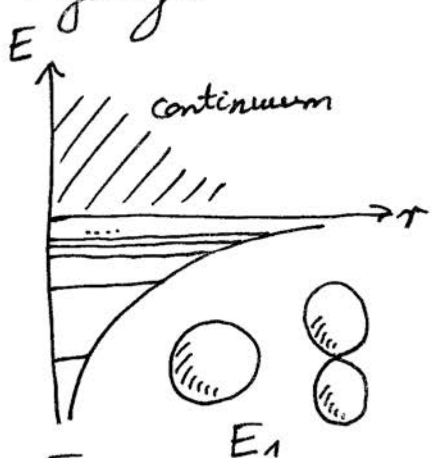


$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

$n = 0, 1, 2, \ldots$
"number of quanta"

(plenty of applications, especially in quantum field theory!)

hydrogen atom



$$E_n = -\frac{E_1}{n^2}$$

$n = 1, 2, 3, \ldots$

$$E_1 = \frac{me^4}{8h^2\varepsilon_0^2} = 13{,}6\,\text{eV}$$

---

many particles: e.g. $\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + V(x_1, x_2)$

with $\Psi(x_1, x_2)$

and so on, for more particles

Impressive Success of QM:

explained...

- atoms & molecules & chemical bonds & crystals
- colors of materials (absorption, heat radiation)
- electrical conductivity, magnetism, mechanical properties
- nuclear structure, radioactivity, fission & fusion, elementary particles
- light quanta

understood/predicted new effects/applications...

- superconductors & superfluids
- lasers
- nuclear & electron spin magnetic resonance
- semiconductors (→ transistors, computers)

future: maybe room-temperature superconductors?
maybe quantum computers?

$>2/3$ of physics research today needs QM directly!

1.2 The meaning of Ψ ?

Compare other wave fields:

Sound waves : pressure p, density ρ

elastic waves, surface waves : displacement field $\vec{u}$

electromagnetic waves : electric field $\vec{E}$ & magnetic field $\vec{B}$ (measure via test charge)

("what is moving?" $\rightarrow$ "aether?" $\rightarrow$ no! relativity!)

(a) Conserved density : probabilities!

Linear wave eq. $\Rightarrow$ expect conserved quantities quadratic in wave field

"local conservation" $\Rightarrow$ find density ρ & current density $\vec{j}$, such that

$$\boxed{\partial_t \rho + \operatorname{div} \vec{j} = 0}$$

"equation of continuity" [hydrodynamics : $\vec{j} = \rho \vec{v}$]

Claim : For the Schrödinger eq.,

$$\rho(\vec{x}) = |\Psi(\vec{x})|^2$$

is a conserved density, with

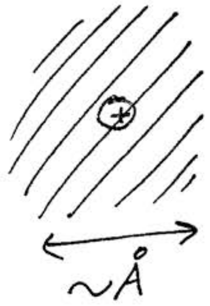
$$\vec{j}(\vec{x}) = \mathrm{Re}\left[\Psi^*(\vec{x}) \underbrace{\frac{-i\hbar\vec{\nabla}}{m}}_{\text{velocity operator}} \Psi(\vec{x})\right]$$

Proof : $\partial_t \rho = \ldots$ use SEQ $\Rightarrow$ ✓

Note : This $\rho, \vec{j}$ are independent of $\hat{H}$

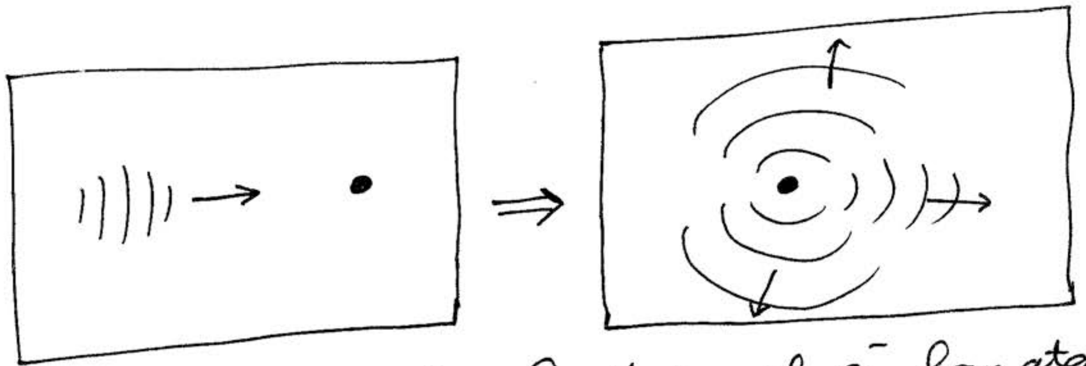
— In contrast to local energy density $\rho_E = \mathrm{Re}\left[\Psi^*(\vec{x}) (\hat{H}\Psi)(\vec{x})\right]$

First guess: $e\cdot|\Psi|^2 =$ charge density of electron, smeared out
(Schrödinger)



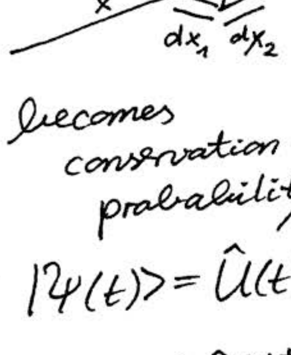
- sounds OK for atoms & molecules: microscopic!
- is used today for molecular structure & motion

... but:
- should different parts of cloud repel each other?
- wave can become extended!



Example (Born): Scattering of e^- from atom (Franck-Hertz experiment)
e^--wave extended over metres!

New interpretation (Born, 1926):
$|\Psi|^2 =$ probability density
$|\Psi(\vec{x})|^2\, dx_1 dx_2 dx_3 =$ probability to find e^- in volume on measurement



$\Rightarrow \quad \partial_t \rho + \mathrm{div}\,\vec{j} = 0$ becomes conservation of probability!

In general: unitary time-evolution $|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle$
with $\hat{U}(t)^+ = \hat{U}(t)^{-1} \Rightarrow$
$$\langle\Psi(t)|\Psi(t)\rangle = \langle\Psi(0)|\,\hat{U}(t)^+\hat{U}(t)|\Psi(0)\rangle = \langle\Psi(0)|\Psi(0)\rangle$$
$\Rightarrow$ global conservation of probability

from Hermitean $\hat{H}$: $\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$ if $\hat{H}$ time-indep.

Statistical ideas before 1926:

- Statistical mechanics:
  Maxwell-Boltzmann distribution,
  Brownian motion, ...
- Radioactive decay
  observe individual stochastic behaviour!
- Einstein's ideas about photons (intensity of light $\hat{=}$ average photon density)

Actual single-quantum measurements at the time:

- In radioactive decay (fluorescence, Geiger-Müller counter)
- cloud & bubble chambers for high-energy particles

showed individual events,
but no quantum interference

Around ~~19~~ 1926, only ensemble measurements were known for quantum phenomena

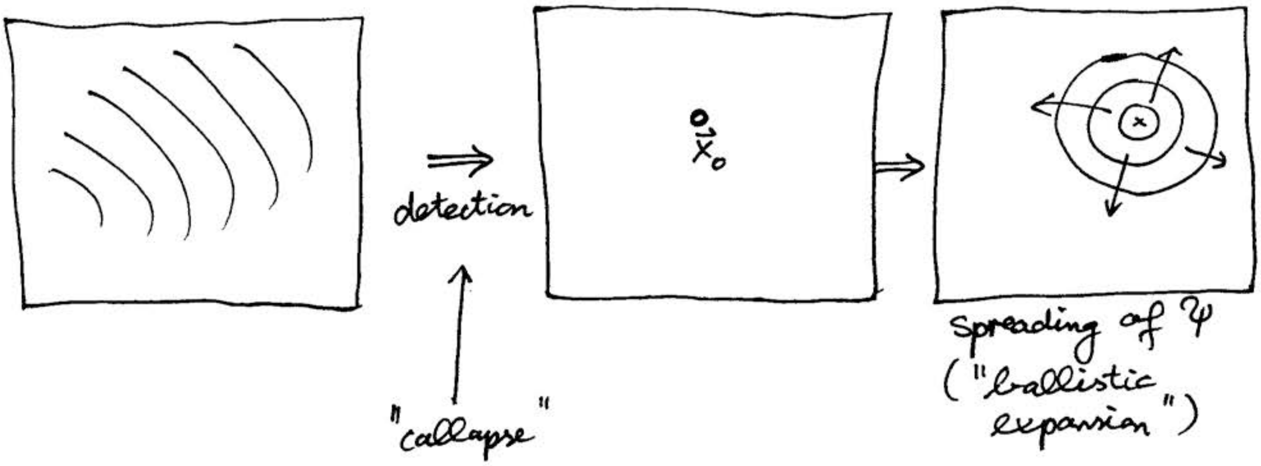
Today ~~Later~~:

- photon detectors (e.g. avalanche photodiode, S.C. photon detectors, ...)
- electron detectors
- also detect single atoms / ions

(b) The "collapse of the wavefunction"

Q: How does Ψ evolve after detection at $\vec{x} = \vec{x}_0$?

A: Replace Ψ by new wavefunction,
localised around $\vec{x}_0$



General rule (von Neumann's projection postulate):

measure observable $\hat{A}$ ($\leftarrow$ hermitean operator)

$\Rightarrow$ obtain eigenvalue A_n with probability

$$|\underbrace{\langle \phi_n}_{\text{eigenstate}} \underbrace{| \Psi \rangle}_{|\Psi\rangle \text{ "projected" onto } |\phi_n\rangle}|^2$$

$\Rightarrow$ "collapse" into new state

$$|\Psi\rangle^{\text{after msmt}} = |\phi_n\rangle$$

Describes most measurements ✓

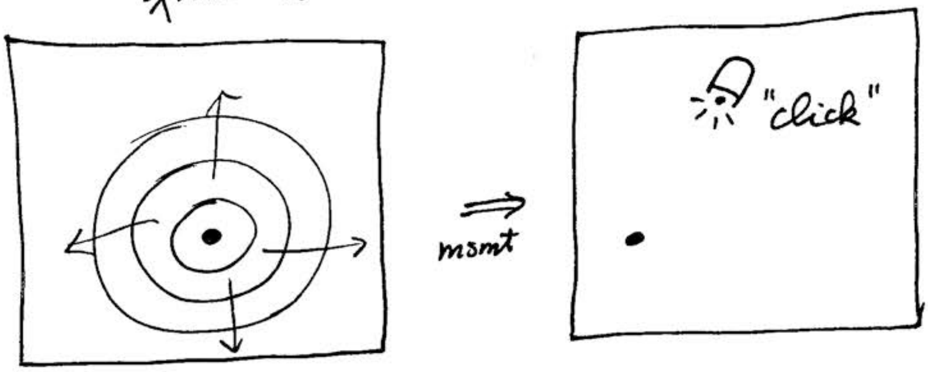
Disadvantages:

- "Ad hoc" postulate, outside of SEQ
- Artificial (?) distinction between quantum system & "classical measurement apparatus"
- Can we describe msmt within SEQ?
- What about msmts with only partial information? ("weak msmts")

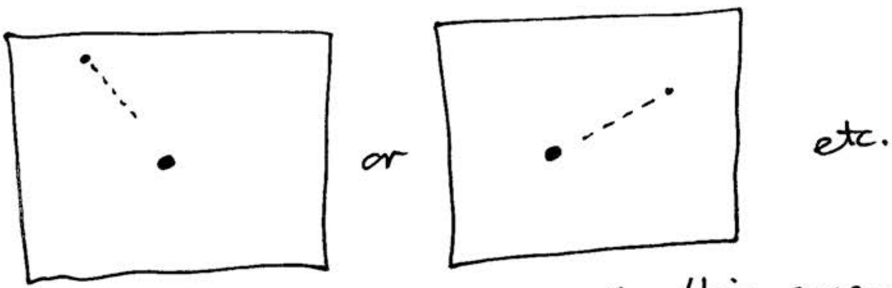
$\longrightarrow$ see modern theory of msmts!

(c) "What happens really at the level of single particles?"

Example: emission of e^- (or photon) from atom



Possible interpretation: emission into a random direction

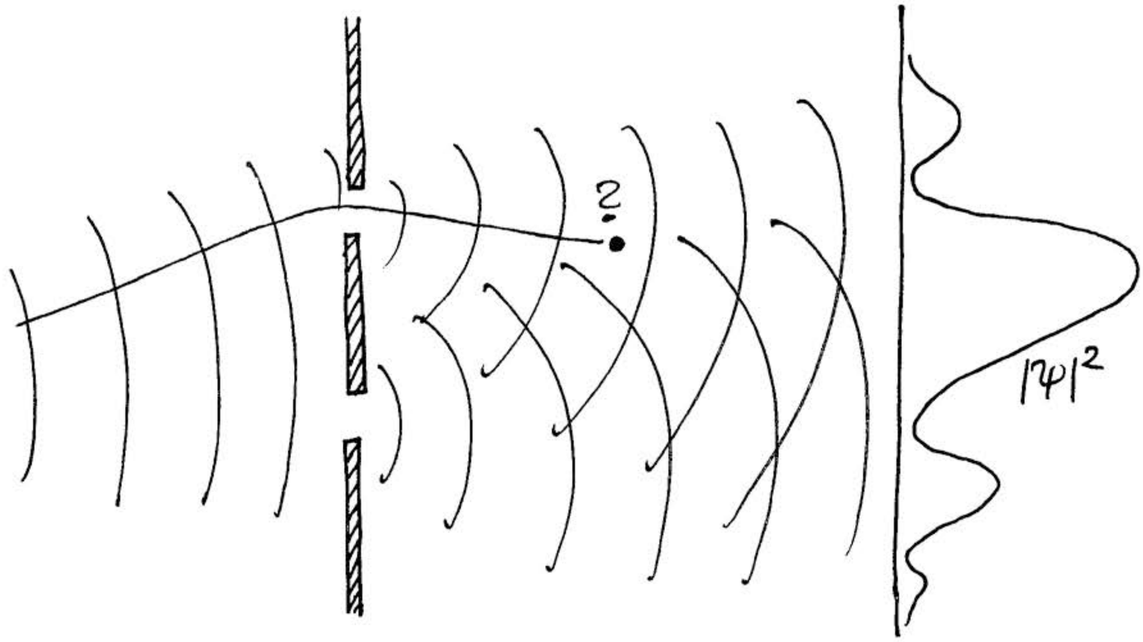


random classical trajectories. OK for this case ✓

($\hat{=}$ Einstein's thinking about photons around 1905)

But: This naive idea does not work for interference setups!

Example: Double-slit setup



If each* e^- has definite trajectory:
each e^- goes through only one slit

$\Rightarrow$ (naive) consequences:

- pattern on screen should be 50% from upper trajectories (e.g.) 50% from lower

- pattern from upper trajs alone can be observed by closing lower slit (and vice versa) *

$\Rightarrow$ contradict experiment: "which-way expt."
  - no interference if any slit closed
  - interference pattern if both slits are open

*as long as there is no mysterious long-range influence!

naive idea yields "$|\psi_u|^2 + |\psi_L|^2$" ← partial wave from lower slit

but we actually observe

$$|\psi_u + \psi_L|^2$$

$$= |\psi_u|^2 + \underbrace{\psi_u^* \psi_L + \psi_L^* \psi_u}_{\text{depends on relative phase}} + |\psi_L|^2$$

~~Heisenberg's explanation~~ "Heisenberg microscope"

- msmt of particle position to accuracy Δx randomizes momentum, with

$$\Delta p \gtrsim \frac{\hbar}{2\Delta x}$$

- if Δx < slit distance $\Rightarrow$ this is enough to destroy interference pattern!

Lesson:
- Observation of quantum particles may strongly perturb their behaviour! (not unexpected for microscopic particles!)
- Perturbation is so strong that we can never observe trajectory without destroying interference effects!

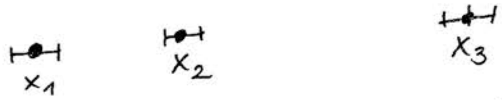
$\Rightarrow$ Copenhagen interpretation (Bohr et al.)

- no trajectories in QM!
- particle position (or momentum etc.) becomes real only upon msmt!

(d) Many-particle wave-functions

$$\psi(x) \longmapsto \psi(x_1, x_2, x_3, \dots, x_N)$$

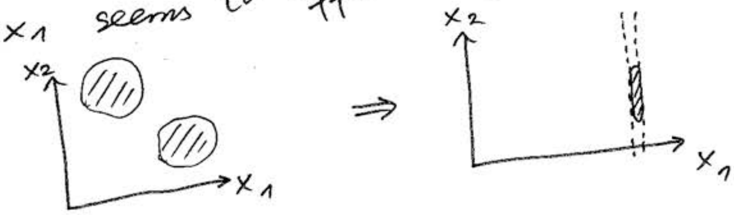
$|\psi(x_1, \dots, x_N)|^2 dx_1 dx_2 \dots dx_N$
= probability to find this configuration



(compare classical statistical physics! $\rho(x_1, \dots, x_N)$)

Challenge for interpretation of ψ:

- Waves in configuration space?
- Measurement of x_1 seems to affect x_2



1.3 Experimental progress in the past 80 years

1925	today
atomic structure: only indirect evidence — frequencies and intensities of transitions	see individual e^- orbitals (AFM, STM) & pictures of atoms & molecules on surfaces
weak excitation of many atoms (e.g. spectroscopy on gas)	excite atoms strongly, 100% in excited state, detect state for single atom, observe quantum jumps
interference experiments only on ensemble of particles (observe intensity)	detect individual quanta (e^-, photons etc.), produce single quanta & do interference expts on those
observe natural quantum systems	design, fabricate, control coherently artificial quantum systems

$> 2/3$ of physics research needs quantum mechanics!

2. Bell's inequalities and entanglement

Is there a theory that "explains" QM?

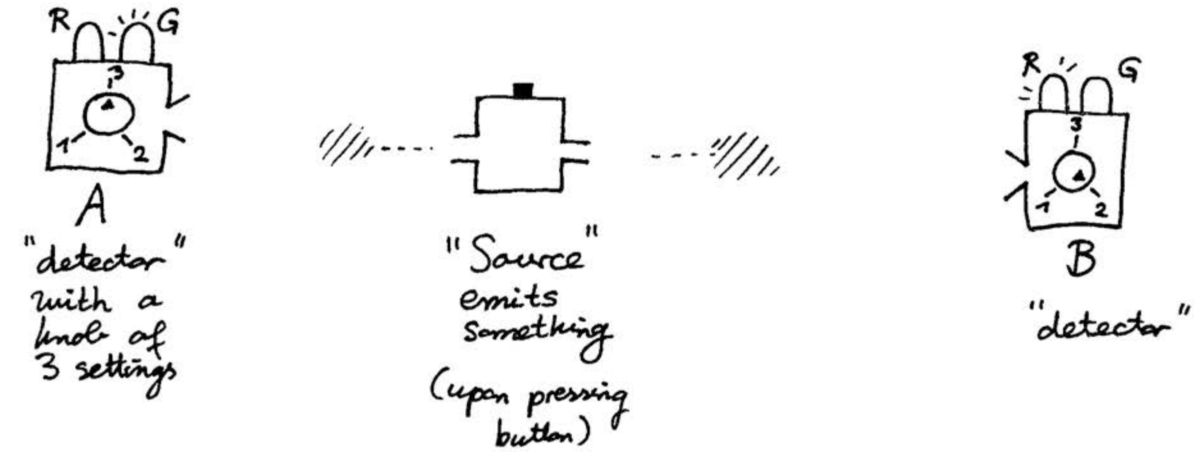
like classical mechanics → classical statistical physics → thermodynamics

2.1 Strange correlations

(following Mermin, 1985)

Idea: Bring out "crazy" aspects of nature w/o reference to quantum theory

"Gedanken-" Experiment:



After emission, each detector flashes one of its lights.

Observations:

(1.) Whenever settings are the same, the same color is found at both detectors. (Even if settings are chosen just at the "last second" before lights flash!)

(2.) If settings are chosen completely randomly & independently $\left[P(11) = P(12) = \ldots = \frac{1}{9} \right]$ then the colors are uncorrelated $\left[P(RR) = P(GG) = P(RG) = P(GR) = \frac{1}{4} \right]$.

Should you be bothered?

(1.) How to get these perfect correlations?

- Not just: "only flash R" $\rightarrow$ would contradict (2.)

- Radio signal or similar between detectors?
  No! If settings are chosen ~~such that~~ just Δt before flash & $c \cdot \Delta t <$ distance $(A-B)$
  Combination of settings not available to detectors!

- Possibly: due to source emitting correlated objects (particles, waves, ...)

  $\Rightarrow$ joint properties!

  Example: Emit identical objects,
  - Detect Shape, Color, Size
  - Detect momenta etc.

~~Each object~~

In each run: each object should know in advance which color to flash for any setting

example:

1	2	3
R	R	G

"instruction set", the same for both objects

Why?
- Cannot know which setting will be chosen, but must show same color if it's the same setting (& no signalling!)
  $\Rightarrow$ need results for all settings
- No randomness allowed at detection, due to (1.)
  (only in choosing instruction set)

Note: This "instruction set" is a property of detector + object
(but separately for A & B)

(2.) If $\boxed{\begin{matrix}1&2&3\\R&R&R\end{matrix}}$ or $\boxed{\begin{matrix}1&2&3\\G&G&G\end{matrix}}$: <u>never</u> RG or GR as results

If anything like $\boxed{\begin{matrix}1&2&3\\R&R&G\end{matrix}}$, $\boxed{\begin{matrix}1&2&3\\G&G&R\end{matrix}}$ etc.:

for random settings $P(RG) = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9} < \frac{1}{4} = \frac{2}{8}$

$\Rightarrow$ ↯ We <u>never</u> get $P(RG) = \frac{1}{4}$

(regardless of instruction sets or random choice thereof)

$\Rightarrow$ (1.) has enforced too much of a tendency towards same colors

Conclusion:

Only reasonable way to guarantee (1.) is incompatible with (2.) !

Ways out:

- Maybe there is no expt. like this? (There is!)
- Superluminal signaling?
  ↳ allows joint instruction sets:

1 1 R R	1 2 G R	2 1 ..	...

2.2 Einstein, Podolsky, Rosen (EPR)

"Is QM the result of some underlying theory?"

Example: Classical mechanics → Statistical Physics (→ Thermodynamics, Hydrodynamics)

Problem: x, p cannot be known simultaneously, "no trajectories" ⇒ ?

Einstein, Podolsky, Rosen (1935): Want to construct situation, where both x and p of a particle can be determined simultaneously with certainty!
(& since QM does not describe these "elements of reality", it is an "incomplete" theory)
How to circumvent Heisenberg's uncertainty principle?
→ Trick: Use two particles, in state

$$\Psi(x_1, x_2) \sim \delta(x_1 - x_2) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \underbrace{e^{i\frac{p}{\hbar}(x_1 - x_2)}}_{\substack{\text{wave function} \\ \text{for } p_1 = +p \\ p_2 = -p}}$$

⇒ (1) $x_1 = x_2$ in any msmt of x_1, x_2
(2) $p_1 = -p_2$ " " p_1, p_2

Now do the following:
measure x_1 ⇒ deduce value that x_2 would have in a msmt (namely: $x_2 = x_1$)

but now get value of p_2 by msmt!

Why possible? (to measure p_2 and still use deduced value of x_2)

Because msmt of x_1 cannot suddenly have changed state of distant particle! (unlike Heisenberg's reasoning for single particle!)

$\Rightarrow$ Both x_2 & p_2 "real"

Summary:

$\boxed{\text{no spooky action at a distance}}$ & $\boxed{\text{QM describes results of expts on 2 particles}}$ $\xrightarrow{\text{EPR}}$ $\boxed{\text{QM "incomplete", more detailed description possible = "hidden variable theory"}}$

Bohr's reply: (my summary)

You are <u>not</u> allowed to treat particles separately, every choice of msmt combinations (like "x_1/p_2", "p_1/p_2" etc.) corresponds to a different experiment, deduction of "what would have been" is not allowed

Schrödinger (1935):

EPR works because

$\psi(x_1, x_2) \neq$ product $\phi_1(x_1) \cdot \phi_2(x_2)$

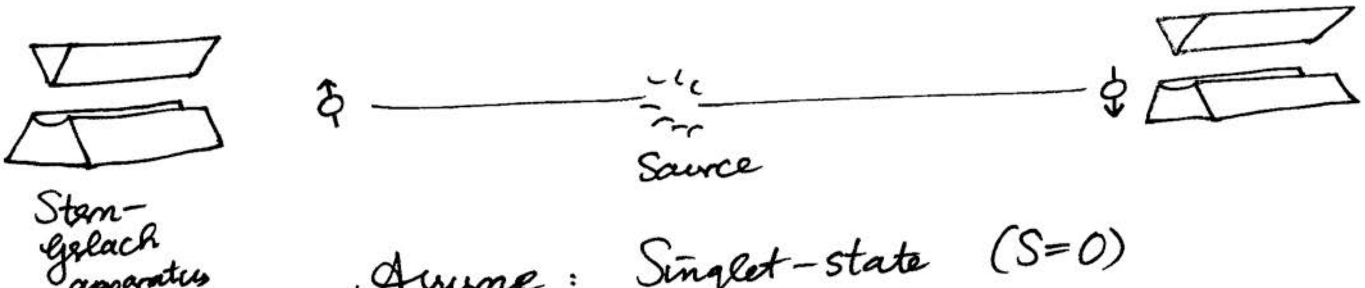
$\Rightarrow$ "ψ is entangled"

["Verschränkung"]

Bohm's version of EPR (1951):

Two spin $\frac{1}{2}$ particles

each: $|\uparrow\rangle$ and $|\downarrow\rangle$ states
(eigenstates of $\hat{S}_z$, with $\hat{S}_z|\uparrow\rangle = +\frac{\hbar}{2}|\uparrow\rangle$ etc.)



Assume: Singlet-state ($S=0$)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$

entangled!

$$\left(\Rightarrow \text{e.g.: } \hat{S}_{z1}|\psi\rangle = \frac{\hbar}{2} \cdot \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 \underset{=}{+} |\downarrow\rangle_1|\uparrow\rangle_2\right)\right)$$

measure Spin 1 along z $\Rightarrow$

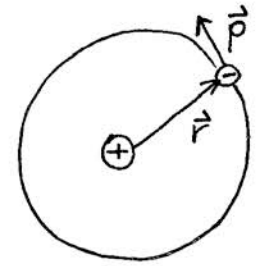
probability 50%: $|\uparrow\rangle_1 \Rightarrow |\psi'\rangle = |\uparrow\rangle_1 |\downarrow\rangle_2$ after measurement

probability 50%: $|\downarrow\rangle_1 \Rightarrow |\psi'\rangle = |\downarrow\rangle_1 |\uparrow\rangle_2$ "

$\Rightarrow$ subsequent msmt of Spin 2 along z:
exactly opposite!

(analogous to "$p_1 = -p_2$")

Reminder: Spin



Orbital angular momentum:

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$$

$\hat{\vec{L}}^2$ has eigenvalues $\hbar^2 \ell(\ell+1)$ $\quad \ell = 0, 1, 2, \ldots$

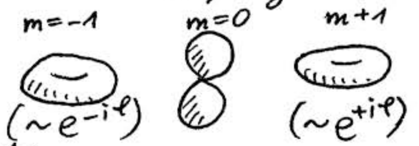
$[\hat{\vec{L}}^2, \hat{L}_{\underset{(x,y)}{z}}] = 0$

$\hat{L}_z$ has eigenvalues $\hbar m$ $\quad m = -\ell, \ldots, +\ell$

$\Rightarrow$ express $\hat{L}_{x,y,z}$ in $2\ell+1$-dim. subspaces $(1, 3, 5, \ldots)$

Example: p-orbitals in hydrogen atom

$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$



This algebra also has representations in even-dim. Hilbert spaces ($\Rightarrow \ell$ = half-integer)

Simplest case: Spin $\frac{1}{2}$ $\quad (\ell = "S" = \frac{1}{2})$

$$\hat{S}_{x,y,z} = \frac{\hbar}{2} \hat{\sigma}_{x,y,z}$$

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli spin matrices (in eigenbasis of $\hat{\sigma}_z$)

Eigenstates of $\hat{\sigma}_z$:

$$\hat{\sigma}_z |\uparrow\rangle = + |\uparrow\rangle$$
$$\hat{\sigma}_z |\downarrow\rangle = - |\downarrow\rangle$$

Eigenstates of $\hat{\sigma}_x$:

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$$
$$|\leftarrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)$$

$$\hat{\sigma}_x |\rightarrow\rangle = +|\rightarrow\rangle$$
$$\hat{\sigma}_y |\leftarrow\rangle = -|\leftarrow\rangle$$

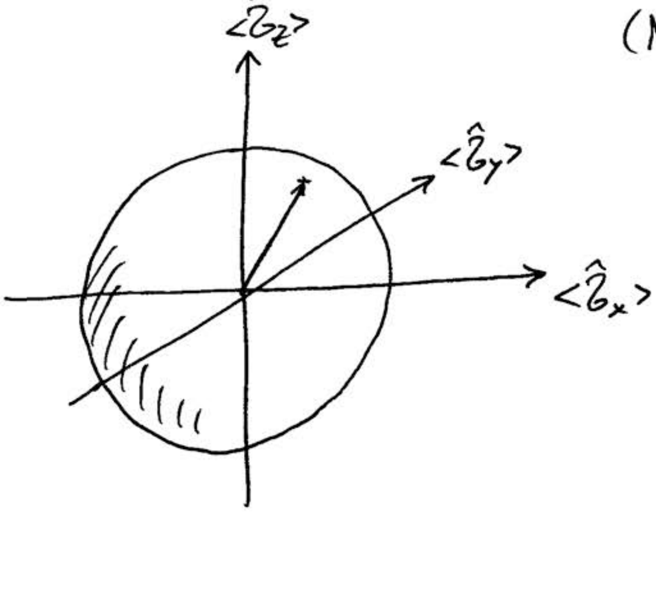
likewise: $\hat{\sigma}_y$ eigenstates

$+1$: $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle + i|\downarrow\rangle\right)$

-1 : $\frac{1}{\sqrt{2}}\left(i|\uparrow\rangle + |\downarrow\rangle\right)$

Spin expectation:

$\langle \hat{\vec{\sigma}} \rangle$ = unit vector for single spin ½ "Bloch vector"



(Note: completely determines state, if we are dealing with single spin ½)

Projection of spin on arbitrary direction:

$$\vec{n}\cdot\hat{\vec{S}} = \frac{\hbar}{2}\,\vec{n}\cdot\hat{\vec{\sigma}}$$

$\vec{n}$: unit vector

$$(\vec{n}\cdot\hat{\vec{\sigma}})^2 = n_x^2 \underbrace{\hat{\sigma}_x^2}_{\mathbb{1}} + \ldots + n_x n_y \underbrace{(\hat{\sigma}_x\hat{\sigma}_y + \hat{\sigma}_y\hat{\sigma}_x)}_{\equiv 0} + \ldots$$

$$= \vec{n}^2 = 1$$

$\Rightarrow$ $\vec{n}\cdot\hat{\vec{\sigma}}$ has eigenvalues ± 1 (just like any spin projection)

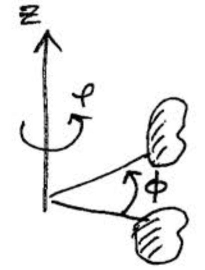
Rotations:

from angular momentum:

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$$

$$\Rightarrow \quad \left[e^{-\phi\frac{\partial}{\partial\varphi}}\psi\right](r,\theta,\varphi) = \psi(r,\theta,\varphi-\phi)$$

rotated by ϕ around z-axis



$$e^{-\phi\frac{\partial}{\partial\varphi}} = e^{-\frac{i}{\hbar}\phi\hat{L}_z} = \text{rotation around } z$$

more general: $e^{-\frac{i}{\hbar}\vec{\phi}\hat{\vec{L}}}$, $\vec{\phi}$ $\parallel$ rotation axis

$[\hat{L}_x, \hat{L}_y] \neq 0$ implies that rotations around different axes do not commute

Generalization to spin: $e^{-\frac{i}{\hbar}\vec{\phi}\hat{\vec{S}}}$

Note:

$$\hat{R}(\phi) = e^{-\frac{i}{\hbar}\phi\frac{\hbar}{2}\hat{\sigma}_z} = e^{-\frac{i}{2}\phi\hat{\sigma}_z} = \mathbb{1}\cdot\cos\left(\frac{\phi}{2}\right) - i\sin\left(\frac{\phi}{2}\right)\cdot\hat{\sigma}_z$$

$\Rightarrow$ full rotation, $\phi = 2\pi$:

$$\hat{R}(\phi=2\pi) = -\mathbb{1}$$

$\rightsquigarrow$ changes sign of spin!
(true for any half-integer spin)

$\rightsquigarrow$ has been measured in experiment!

~~Multiple~~ Two Spin $\frac{1}{2}$:

Product Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

Product basis, e.g.: $|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 \equiv |\uparrow\downarrow\rangle$

$$\hat{S}_{z1}|\uparrow\downarrow\rangle = (\hat{S}_{z1}\otimes\mathbb{1}_2)|\uparrow\downarrow\rangle$$
$$= +\frac{\hbar}{2}|\uparrow\downarrow\rangle$$

$$\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2 \quad \text{total spin}$$

$$\hat{\vec{S}}^2 = \hat{\vec{S}}_1^2 + \hat{\vec{S}}_2^2 + 2\hat{\vec{S}}_1\cdot\hat{\vec{S}}_2$$

Eigenstate for $\hat{\vec{S}}^2$ with eigenvalue 0:

"Singlett" state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

$$\hat{\vec{S}}|\psi\rangle = 0$$

$$\Rightarrow e^{-\frac{i}{\hbar}\vec{\phi}\hat{\vec{S}}}|\psi\rangle = |\psi\rangle$$

rotationally invariant!

Measure $\hat{S}_x$ for Spin 1 & 2 $\Rightarrow$

$$P_{\rightarrow\rightarrow} = |\langle \rightarrow_1 \rightarrow_2 | \psi \rangle|^2$$

We have: $|\rightarrow_1\rightarrow_2\rangle = \frac{1}{2}\left(|\uparrow_1\uparrow_2\rangle + |\uparrow_1\rangle|\downarrow_2\rangle + |\downarrow_1\rangle|\uparrow_2\rangle + |\downarrow_1\rangle|\downarrow_2\rangle\right)$

$$\langle \rightarrow_1 \rightarrow_2 | \psi \rangle = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} (1 - 1) = 0$$

$$P_{\rightarrow\leftarrow} = |\langle \rightarrow_1 \leftarrow_2 | \psi \rangle|^2$$

We have: $|\rightarrow_1\leftarrow_2\rangle = \frac{1}{2}\left(|\uparrow_1\uparrow_2\rangle - |\uparrow_1\rangle|\downarrow_2\rangle + |\downarrow_1\rangle|\uparrow_2\rangle - |\downarrow_1\rangle|\downarrow_2\rangle\right)$

$$\langle \rightarrow_1 \leftarrow_2 | \psi \rangle = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot (-1 - 1) = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow P_{\rightarrow\leftarrow} = \frac{1}{2}$$

likewise $P_{\leftarrow\rightarrow} = \frac{1}{2}$ and $P_{\leftarrow\leftarrow} = 0$

$\Rightarrow$ S_x measurement results are <u>also</u> exactly opposite! (just like S_z)

Alternative approach: project onto eigenstates subsequently

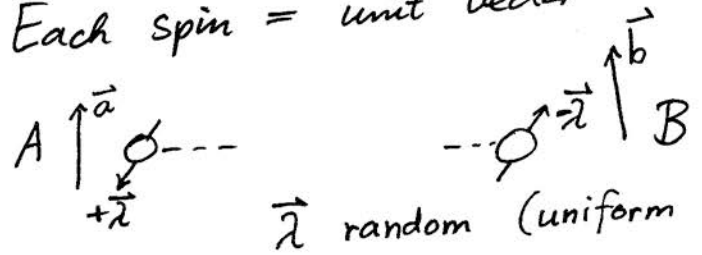
$$|\psi\rangle \rightarrow \underbrace{|\rightarrow_1\rangle\langle\rightarrow|_1 \psi\rangle}_{= |\rightarrow_1\rangle \cdot \frac{1}{\sqrt{2}} |\leftarrow_2\rangle} \rightarrow |\rightarrow_2\rangle\langle\rightarrow|_2 |\rightarrow_1\rangle\langle\rightarrow|_1 \psi\rangle$$

2.3 Bell's inequalities

Q: Can we find underlying theory to explain EPR-expt. (in Bohm's version)?

First, naive, attempt:

Each spin = unit vector



$\vec{\lambda}$ random (uniform over sphere)

Msmt on A, along direction $\vec{a}$

$\Rightarrow$ Result "$\uparrow$" if $\vec{\lambda}\cdot\vec{a} > 0$

$$\underbrace{A(\vec{a},\vec{\lambda})}_{\text{msmt result: } \pm 1 \text{ for } \uparrow/\downarrow} = \text{sign}\,(\vec{\lambda}\cdot\vec{a})$$

likewise:

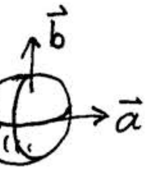
$$B(\vec{b},\vec{\lambda}) = -\text{sign}\,(\vec{\lambda}\cdot\vec{b})$$

$\Rightarrow$ If $\vec{a}=\vec{b}$, then <u>always</u>

$$A(\vec{a},\vec{\lambda}) = -B(\vec{a},\vec{\lambda})$$

$\Rightarrow$ Opposite spin directions ✓ just like QM!

If $\vec{a}\perp\vec{b}$: $A\cdot B>0$ as often as $A\cdot B<0$



$$\Rightarrow \langle AB\rangle = \int d\vec{\lambda}\; \rho(\vec{\lambda}) A(\vec{a},\vec{\lambda}) B(\vec{b},\vec{\lambda}) \equiv 0$$

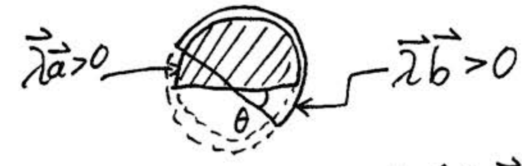
no correlations, like QM! ✓

Bell's important new idea:

Check also ~~$\vec{a} \parallel \vec{b}$ and $\vec{a} \perp \vec{b}$~~

other angles between $\vec{a}$ and $\vec{b}$!

In this model:



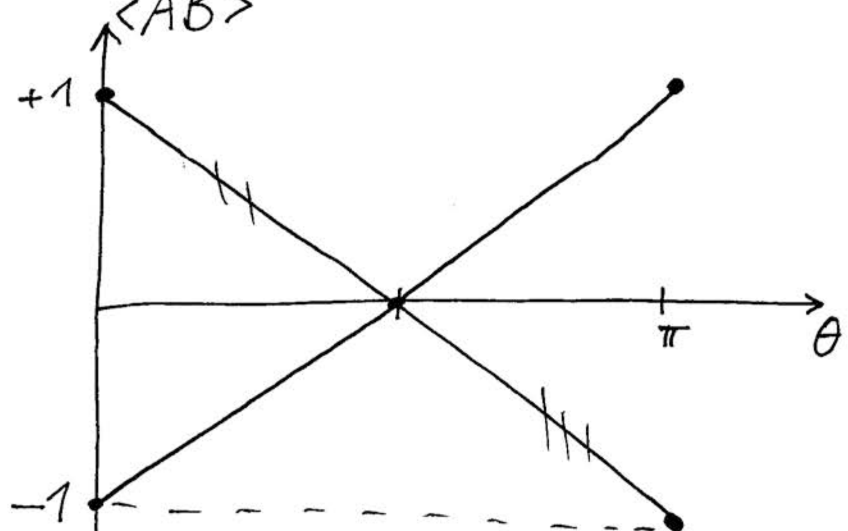
$$\theta = \sphericalangle(\vec{a}, \vec{b})$$

$\text{sign}(\vec{\lambda}\cdot\vec{a}) \neq \text{sign}(\vec{\lambda}\cdot\vec{b})$ in a fraction

$\frac{2\theta}{2\pi} = \frac{\theta}{\pi}$ of cases

$$\Rightarrow \quad \langle AB \rangle = (+1)\cdot\frac{\theta}{\pi} + (-1)\cdot\left(1 - \frac{\theta}{\pi}\right)$$

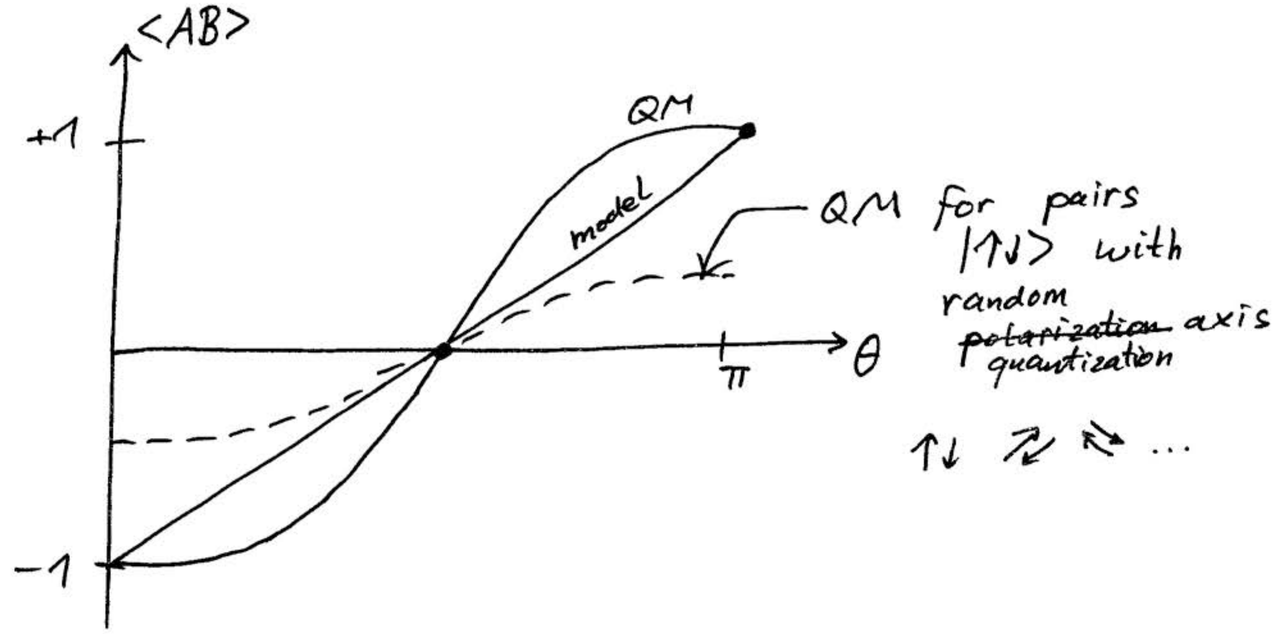
$$= -\left(1 - 2\frac{\theta}{\pi}\right) \qquad \text{(for negative } \theta\text{: } \theta \mapsto |\theta|)$$



Compare QM:

$$\langle AB\rangle\Big|_{\substack{QM,\\ \text{singlet state}}} = \langle\psi|(\vec{a}\cdot\hat{\vec{\sigma}}_1)(\vec{b}\cdot\hat{\vec{\sigma}}_2)|\psi\rangle$$

$$= -\vec{a}\cdot\vec{b} = -\cos\theta$$



$\Rightarrow$ If we reproduce perfect correlations, we fail for other angles!

Is this a general rule?

Define a general

<u>local hidden variable</u> (LHV) <u>model</u>

for the EPR(Bohm) – Experiment:

Hidden variable(s) λ
Probability density $\rho(\lambda)$
Measurement results $A(\vec{a},\lambda) \in \{+1,-1\}$
and $B(\vec{b},\lambda) \in \{+1,-1\}$
for detector settings $\vec{a}, \vec{b}$ (= msmt axis)

<u>Local</u> because we do <u>not</u> allow
$$A(\vec{a},\vec{b},\lambda)$$

$\Rightarrow$ Constraints for statistics?
Consider $\langle AB \rangle = \int d\lambda\, \rho(\lambda)\, A(\vec{a},\lambda) B(\vec{b},\lambda) \equiv E(\vec{a},\vec{b})$
Look at several combinations of settings
($\rightarrow$ both for $\vec{a} \parallel \vec{b}$ and other angles!)
& get joint constraint!

Bell's original version: 2 directions at each spin $(\vec{a},\vec{a}'; \vec{b},\vec{b}')$
Let $A(\vec{a}',\lambda) \equiv A'$ etc.

Let $+\vec{a}' = \vec{b}'$ and assume perfect
(anti-)correlations are observed $\Rightarrow$

$$A' = -B'$$
for each λ

(*for "almost all" λ,
i.e. except for a
set of measure zero)

$$|AB' - AB|$$
$$= |A(B'-B)| \underset{A=\pm 1}{=} |B'-B| \underset{B'=-A'}{=} |A'+B| \underset{\substack{A'=\pm 1\\ B=\pm 1}}{=} 1 + A'B$$

Now $\langle |X| \rangle \geq |\langle X \rangle| \Rightarrow$

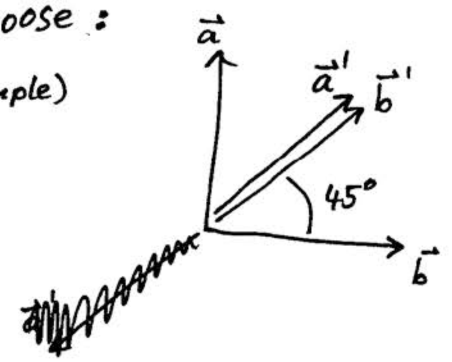
$$\boxed{|\langle AB' \rangle - \langle AB \rangle| \leq 1 + \langle A'B \rangle}$$

Bell's inequality 1964

Obeyed by <u>every</u> LHV (that shows perfect anticorrelations for $-\vec{a}' = \vec{b}'$)

Compare with QM, $\langle AB \rangle = -\vec{a}\cdot\vec{b}$ etc.:

Choose: (for example)



$$\Rightarrow \langle AB' \rangle_{QM} = -\frac{1}{\sqrt{2}}, \quad \langle AB \rangle = 0, \quad \langle A'B \rangle = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow |\langle AB' \rangle - \langle AB \rangle| = \frac{1}{\sqrt{2}}$$
$$1 + \langle A'B \rangle = 1 - \frac{1}{\sqrt{2}}$$

But
$$\frac{1}{\sqrt{2}} \nleq 1 - \frac{1}{\sqrt{2}}$$

because $1 \nleq \sqrt{2} - 1 \approx 0.4$

Generalization (Clauser, Holt, Horne, Shimony 1969)

$$|AB + AB'| + |A'B - A'B'|$$

A, B, A', B' four random variables, with values $0, \pm 1$

$$\leq |B + B'| + |B - B'|$$

$$\leq 2$$

if only values ± 1 & 0 — useful for "no detection" results (finite detector efficiency)

& $\langle |X| \rangle \geq |\langle X \rangle|$

$$\Rightarrow \boxed{|\langle AB \rangle + \langle AB' \rangle| + |\langle A'B \rangle - \langle A'B' \rangle| \leq 2}$$

CHHS inequality
for any LHV
with $A \in \{0, +1, -1\}$ etc.
and no further assumptions
(don't need $B' = -A'$ etc.)

Note: <u>Locality</u> is there because the value of $A = A(\vec{a}, \lambda)$ is assumed to be <u>independant</u> of the other setting, i.e.

$$\text{"}AB\text{"} = A(\vec{a}, \lambda)\, B(\vec{b}, \lambda)$$
$$\text{"}AB'\text{"} = A(\vec{a}, \lambda)\, B(\vec{b}', \lambda)$$

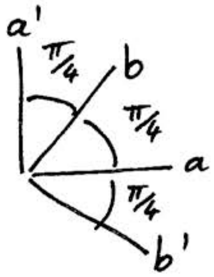
the same A!

CHHS is the version used in modern analysis!

CHHS:

Compare with QM (for spin singlet)

Choose



$$\Rightarrow \quad \langle AB \rangle = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\langle AB' \rangle = -\cos\left(\frac{3\pi}{4}\right) = +\frac{1}{\sqrt{2}}$$

$$\langle A'B \rangle = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\langle A'B' \rangle = -\cos\left(3\frac{\pi}{4}\right) = +\frac{1}{\sqrt{2}}$$

$\Rightarrow$ lhs of CHHS:

$$\frac{4}{\sqrt{2}} = 2\sqrt{2} \approx 2.8 \overset{!}{>} 2$$

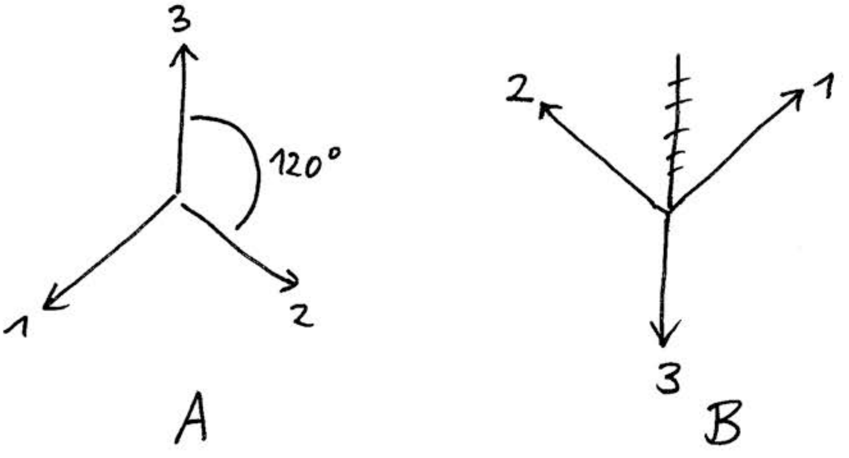
$\Rightarrow$ ↯ conflict between QM & LHV !

$\Rightarrow$ Test in experiments!

Note: Mermin's "gedanken"-experiment

'Same settings → same colors'
'Random " → random colors'

Works for spin $\frac{1}{2}$ if settings $\hat{=}$ msmt axis in the following way:

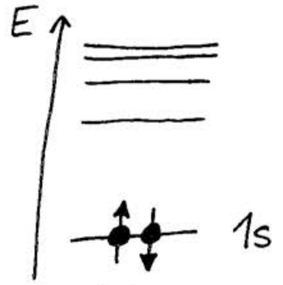


(→ exercises!)

2.4 Bell test experiments (for EPR/Bohm)

Overview: Possible systems

He-atom



ground state:
spin singlet of two e^-
(fermions ⇒ symmetric orbital wave fct.)

in principle: double ionization
⇒ singlet pair

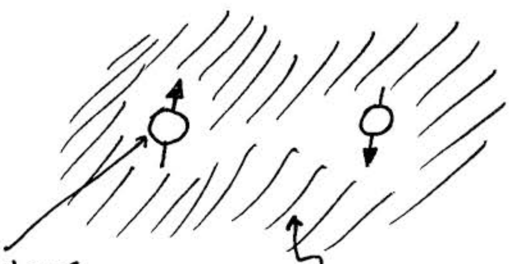
similar e^--singlet states:
other atoms, quantum dots 
or even Fermi sea (in metal)

$\}$ $S=0$ for even number of e^-

Cooper pairs in a superconductor

(Note: need careful extraction of singlet pairs out of many-body state!)

Nuclear spins in molecules



e.g. H_2
proton spins in singlet
= "para-hydrogen"

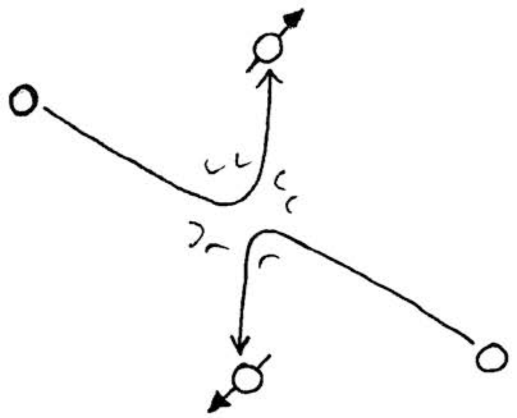
triplet = "ortho-hydrogen"

"para" preferred at low T

difference in rotational states (protons need total antisymmetric Ψ!)

modern version on Hg_2 (mercury dimer)
→ in progress
[^{199}Hg has only nuclear spin $\frac{1}{2}$]

Singlets from scattering



Two fermions (spin 1/2) ⇒
total Ψ antisymmetric under particle exchange!
~~spin~~

⇒

spin part	orbital part
singlet $S=0$ $\frac{1}{\sqrt{2}}(\vert\uparrow\downarrow\rangle - \vert\downarrow\uparrow\rangle)$	symmetric
triplet $S=1$ $\vert\uparrow\uparrow\rangle$ or $\vert\downarrow\downarrow\rangle$ or $\frac{1}{\sqrt{2}}(\vert\uparrow\downarrow\rangle + \vert\downarrow\uparrow\rangle)$	anti-symmetric $\Rightarrow \Psi_{orbital}(x_1=x, x_2=x) = 0$

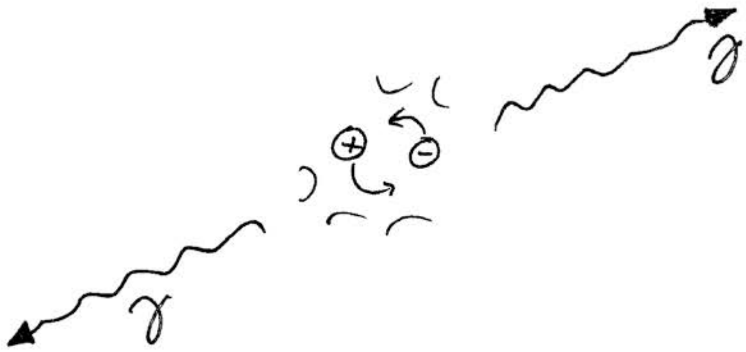
⇒
for "short-range"* interaction:
mostly singlet state is scattered!
*compared to λ!

In general: uncorrelated spins before scattering
→ correlated spins after "
(depending on scattering angle)

→ "low energy"* proton-proton scattering expts

*~MeV

Photons from positronium annihilation



Photon "spin" = polarization of electromagnetic wave
e.g. horizontal $|H\rangle$, vertical $|V\rangle$

Result for this case:

$$\frac{1}{\sqrt{2}}\left(|HV\rangle - |VH\rangle\right)$$

alternatively: circular polarization
right-handed $|R\rangle$, left-handed $|L\rangle$

$$\frac{1}{\sqrt{2}}\left(|RL\rangle - |LR\rangle\right)$$

zero total spin

→ Wu & Shaknov 1950
experiment

Note:

Polarization of light

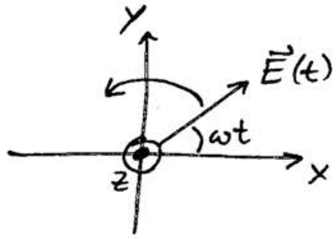
(first, classical):

Consider

$$\vec{E}(t) = \mathrm{Re}\left[\vec{E}_0 \, e^{i(\vec{k}\vec{r} - \omega t)}\right]$$

$$\text{e.g. } \underset{\text{at } \vec{r}=0}{\vec{E}_0} = \begin{pmatrix} 1 \\ i \end{pmatrix} \Rightarrow \vec{E}(t) = \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}$$

$\Rightarrow$



let's call this "right-handed"

(Warning: opposite definitions are also used!)

$$\Rightarrow \text{QM: let } |R\rangle = \frac{|H\rangle + i|V\rangle}{\sqrt{2}}$$

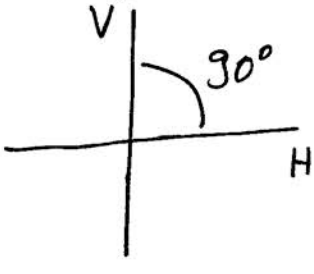
$$|L\rangle = \frac{|H\rangle - i|V\rangle}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}}\left(|RL\rangle - |LR\rangle\right) = \ldots = \frac{i}{\sqrt{2}}\left(|VH\rangle - |HV\rangle\right)$$

Note:

spin 1/2

light



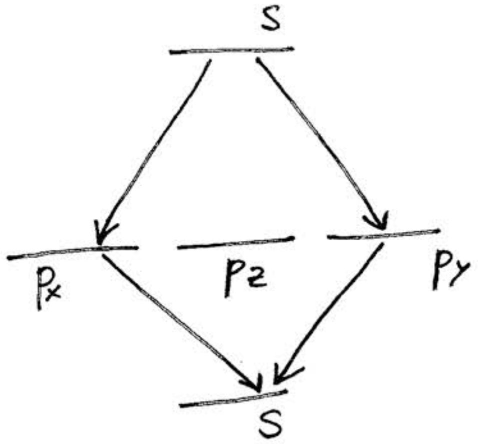
$$\langle AB\rangle = -\cos\sphericalangle(\vec{a},\vec{b})$$

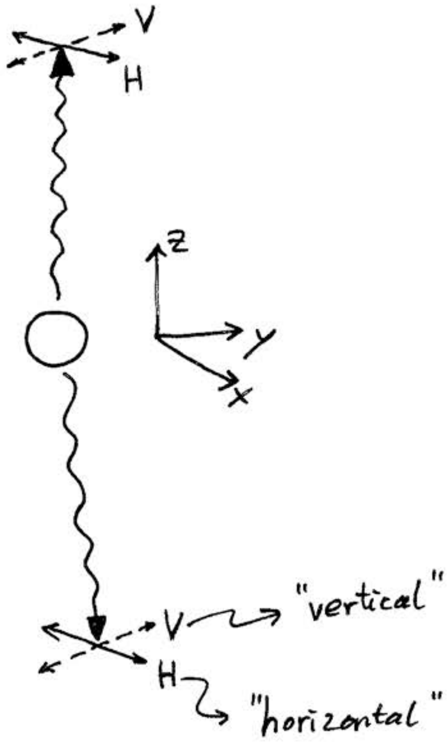
for $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

$$\langle AB\rangle = -\cos 2\sphericalangle(\vec{a},\vec{b})$$

for $\frac{1}{\sqrt{2}}\left(|HV\rangle - |VH\rangle\right)$

Photons from atomic cascade





Simple picture:
in an $S \leftrightarrow p_x$ transition,
the dipole moment oscillates
along x:

$$\langle p_x | \hat{\vec{r}} | s \rangle \parallel x\text{-axis}$$

for single e^-
(otherwise: $\sum_j \hat{\vec{r}}_j$)

here: coherent superposition of both pathways

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|HH\rangle + |VV\rangle \right)$$

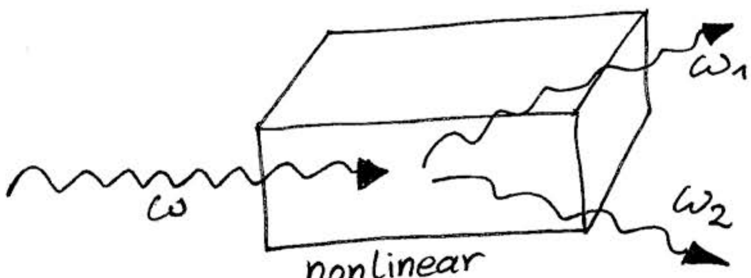
$|\psi\rangle$ is independent of choice for
linear polarization axis:

basis transformation: $|H\rangle = c\,|H'\rangle + s\,|V'\rangle$ where $c = \cos\theta$
$|V\rangle = c\,|V'\rangle - s\,|H'\rangle$ $s = \sin\theta$

$$\Rightarrow |\psi\rangle = \ldots = \frac{1}{\sqrt{2}} \left(|H'H'\rangle + |V'V'\rangle \right)$$

[Note: "Local" unitary transformation applied to photon 2 can turn this into singlet state-form $|H_2\rangle \mapsto |V_2\rangle$, $|V_2\rangle \mapsto -|H_2\rangle$]

Photons from parametric down-conversion



split photon into two

nonlinear material:

$\hat{P} \sim .. E + .. E^2 + ... E^3 + \dots$

(polarization)

energy conservation: $\hbar\omega = \hbar\omega_1 + \hbar\omega_2$

momentum conservation: $\hbar\vec{k} = \hbar\vec{k}_1 + \hbar\vec{k}_2$

Quantum description:

$$\hat{H} = \sum_k \hbar\omega_k \underbrace{\hat{a}_k^+ \hat{a}_k}_{\substack{\text{photon} \\ \text{number} \\ \text{in mode } k}} + \sum_{k,k_1,k_2} \hbar \underbrace{g_{k,k_1,k_2}}_{\substack{\text{matrix} \\ \text{element} \\ \text{for transition,} \\ \text{includes} \\ \text{constraint} \\ \vec{k} = \vec{k}_1 + \vec{k}_2}} \overbrace{\hat{a}_{k_1}^+ \hat{a}_{k_2}^+}^{\text{create pair!}} \overbrace{\hat{a}_k}^{\substack{\text{destroy} \\ \text{incoming} \\ \text{photon}}} + \text{h.c.}$$

$(+\dots)$

$$k = (\underbrace{\vec{k}}_{\substack{\text{wave} \\ \text{vector} \\ \text{(momentum)}}}, \underbrace{\sigma}_{\substack{\text{polarization} \\ \text{index (two values)}}})$$

entanglement in:

- energy
- momentum
- polarization

  (depends on details of material)

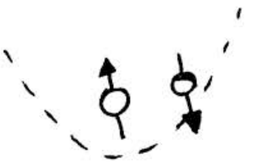
often: select energy & directions $\Rightarrow$ keep only polarization entanglement

e.g. $\frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle)$

its

l control : manipulate &
prepare entangled state
via pulses & interaction

xamples:

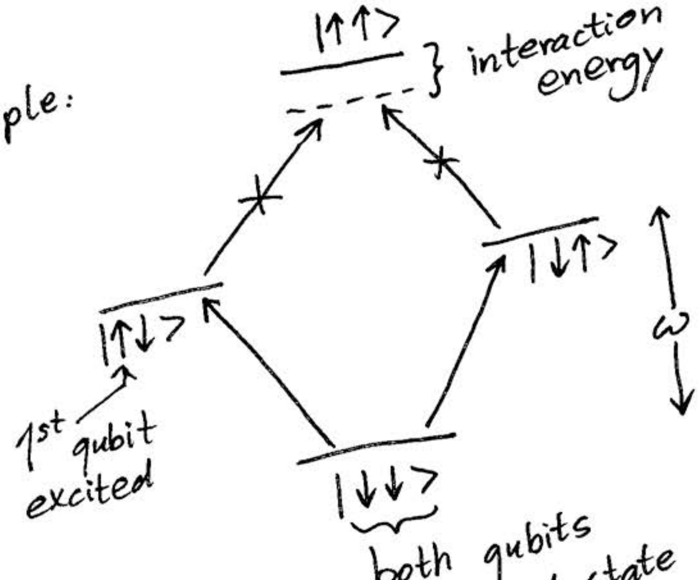


trapped ions

Superconducting qubits

Example:

$[\hat{H} \sim \hat{\sigma}_{z1}\hat{\sigma}_{z2}]$



switch on
drive at ω
for well-defined
time $\Rightarrow$
produce
$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$

(no term with $|\uparrow\uparrow\rangle$)

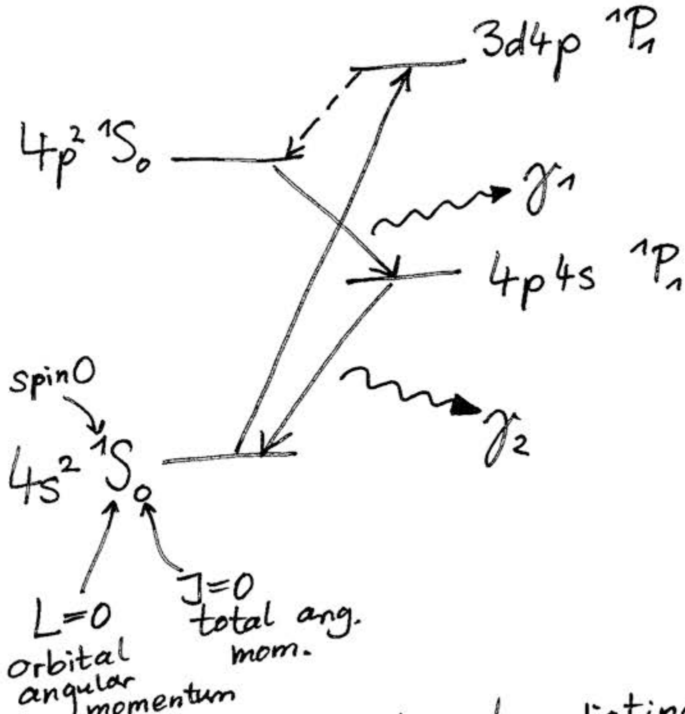
Other example:
Interaction $|\uparrow\downarrow\rangle \leftrightarrow |\downarrow\uparrow\rangle$ $[\hat{H} \sim \hat{\sigma}_{x1}\hat{\sigma}_{x2}]$
$\Rightarrow$ Excite 1st qubit, wait $\Rightarrow$ $|\psi\rangle = \frac{|\uparrow\downarrow\rangle - i|\downarrow\uparrow\rangle}{\sqrt{2}}$

History

- positronium decay : 1950 & 70s
- proton scattering (Lamehi-Rachtig & Mittig '76)
- First cascade experiments in 70s

1972 Freedman & Clauser (first one)

Calcium



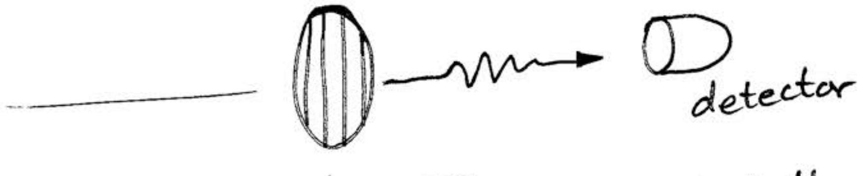
→ lifetime ~5nsec

⇒ γ_1 & γ_2 emitted within this time!

⇒ coincidence counting!

Frequency filters to distinguish γ_1/γ_2

Polarization filters:

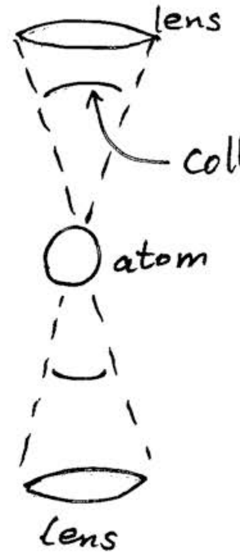


only one polarization detected!

Overall Detector efficiency $\sim 10^{-3}$ (including solid angle)

Dark count rate ~ 100 Hz

Collect only "back-to-back" photon pairs



(if too large: spoil polarization correlations)

$\Rightarrow$ Coincidence rate $\gtrsim 0.1$ Hz
accidental " $\lesssim 0.01$ Hz

~ 200 h data collection!

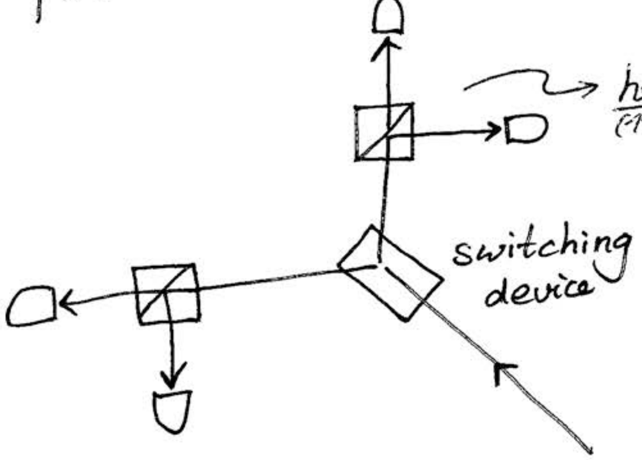
After accounting for inefficiencies:
- agree with QM
- violate 'suitably modified' Bell inequality

(deviation $\approx$ "4 standard deviations")

Aspect et al., 1980s

Ca cascade

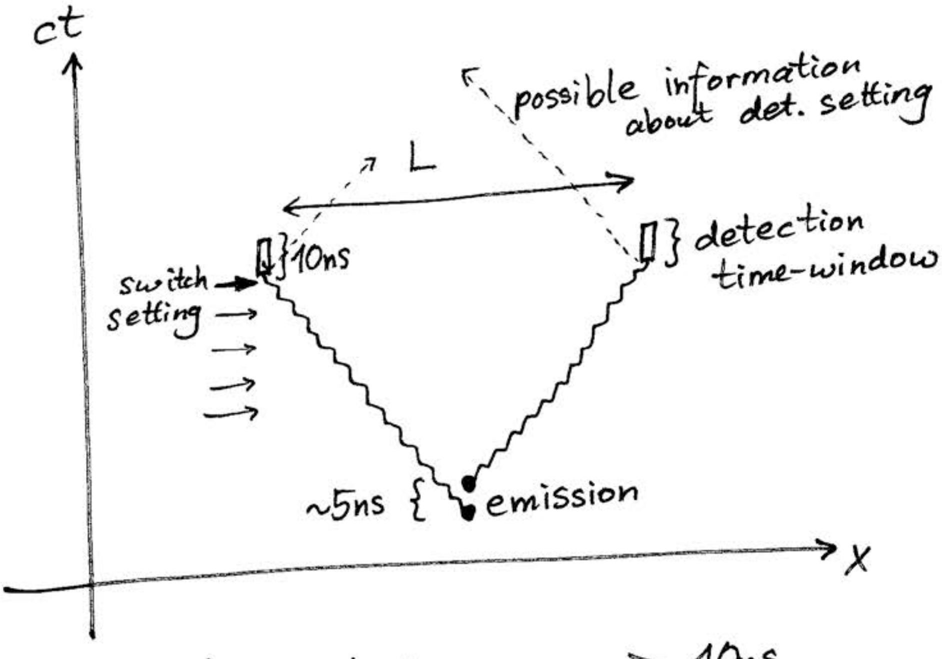
- first to use <u>both</u> polarization channels
- first to switch fast the polarizer direction 1982



→ here: rather only one polarization filter (1982)

(Bragg reflection on ultrasound wave in water
→ periodic switching at ~ 50 MHz)

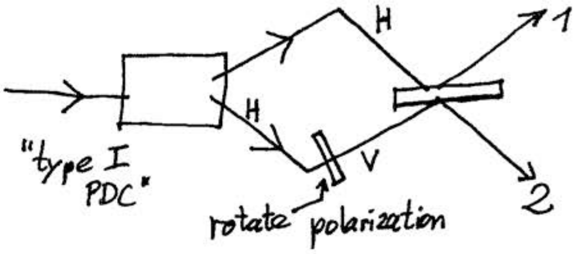
⇒ "space-like" separation of detection



here: $L/c \sim 40\,\text{ns} > 10\,\text{ns}$

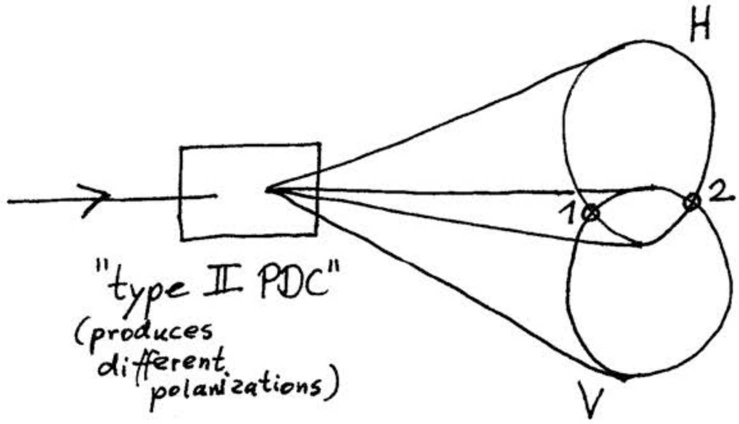
Parametric down-conversion experiments

1980s: Alley & Shih,
Hong, Ou, Mandel, ...



$$|\psi\rangle = (r|1_H\rangle + t|2_H\rangle)(r|2_V\rangle + t|1_V\rangle)$$
$$= \ldots + \underbrace{r^2|1_H 2_V\rangle + t^2|2_H 1_V\rangle}_{\text{"post-"select only these events}}$$

1995 Kwiat et al. (Zeilinger group)
New source



(shown here: only light at one frequency)

Along directions 1,2:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|H_1, V_2\rangle + e^{i\alpha}|V_1, H_2\rangle\right)$$

(without post-selection etc.!)

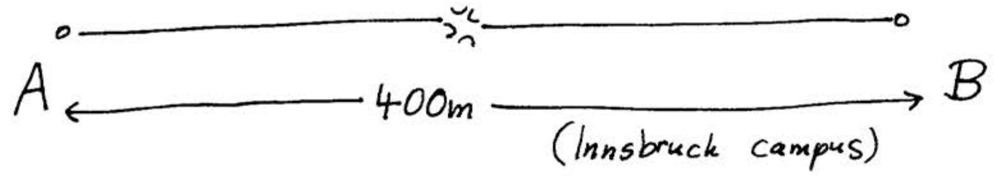
"100 standard deviations violation" in <5min

Coincidence rates > kHz

>10% detection efficiency

1998 Ideal locality conditions:
Weihs et al. (Zeilinger group, Innsbruck)

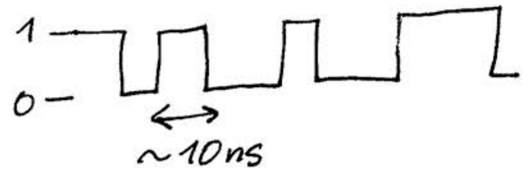
Space-like separation
independent <u>random</u> switching of settings

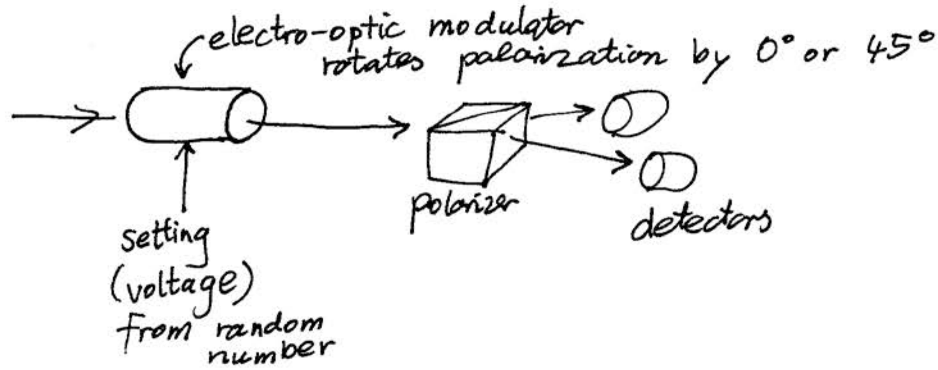


$\frac{400\text{m}}{c} \approx 1.3\,\mu s$

$\Rightarrow$ choice of setting & detection need to occur in $< 1.3\,\mu s$

Physical random number generator:





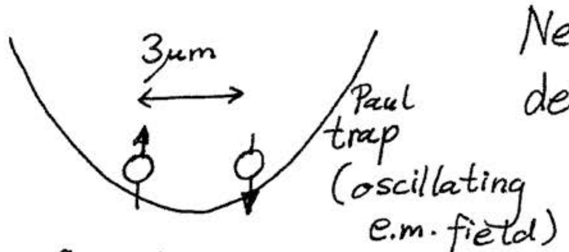
total time ≤ 100 ns !

detectors: collection + detection efficiency 5%
~10 kHz signal counts
~ few 100 Hz dark counts

time-tag detection events : independent atomic clocks
$\Rightarrow$ compare later on computer, extract statistics
$\rightarrow$ <u>30 std violation</u>!

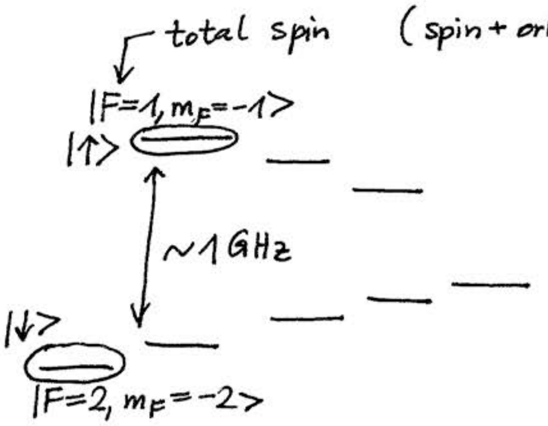
Ion trap Bell experiment

Wineland group 2001



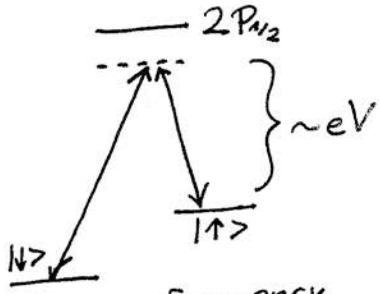
$^9Be^+$ ions

Nearly perfect detection efficiency!



level scheme
(hyperfine levels, all part of "ground state" $2S_{1/2}$)

Control state via "stimulated Raman transition" (two lasers)



[Note: $2P_{1/2}$ should be below - - - -]

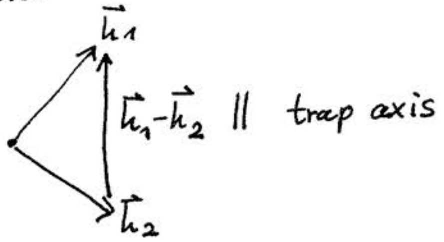
frequency difference of lasers $= \omega_\uparrow - \omega_\downarrow$

Coupling to motion during transitions:

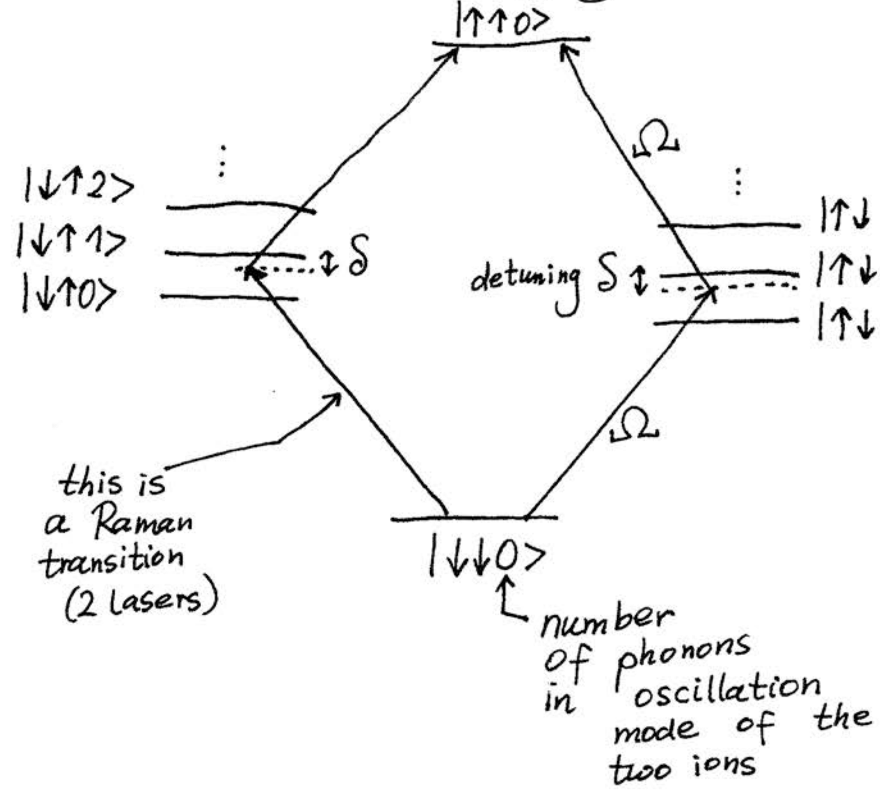
laser drive: $\hat{H} \sim |e\rangle\langle g| \underbrace{e^{ik(\bar{x}+\delta\hat{x})}}_{\approx e^{ik\bar{x}}(1+ik\delta\hat{x}+\ldots)} + h.c.$

$$\delta\hat{x} = \sum \text{normal modes in trap}$$

Note: for Raman transition: consider $\Delta\vec{k} = \vec{k}_1 - \vec{k}_2$



Molmer/Sorensen entangling gate:



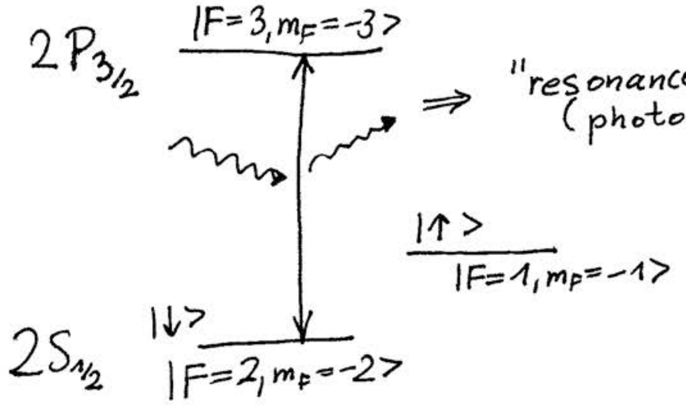
"two-photon transition"

$$\Omega_{eff} = \frac{\Omega^2}{\delta}$$

$\Rightarrow$ after appropriate time: create

$$\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle - i\,|\downarrow\downarrow\rangle\right)$$

Detection:



$\Rightarrow$ "resonance fluorescence" (photon scattering by ion)

$|\downarrow\rangle \Rightarrow \sim 60$ photons detected (⇒ photo-~~detector~~ detection efficiency not important)

$|\uparrow\rangle \Rightarrow$ dark

Different polarizer settings:
Rotation prior to detection, via Rabi pulse connecting $|\uparrow\rangle$ & $|\downarrow\rangle$

$\Rightarrow$ "Detection loophole" closed, but "Locality loophole" open