7. Geometrical phases

7.1 Aharonov-Bohm effect

Classical motion of a charged particle:

electric field ... magnetic field

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \quad [SI]$$

$\vec{E}, \vec{B}$ fulfill Maxwell's eqs, especially

$$\operatorname{div}\vec{B} = 0 \quad , \quad \operatorname{rot}\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

[homogeneous eqs.]

$\Rightarrow$ can be expressed via

$$\vec{B} = \operatorname{rot}\vec{A}\,, \quad \vec{E} = -\partial_t\vec{A} - \vec{\nabla}U$$

vector potential ... electrostatic potential (with $V = qU$)

$\vec{A}, U$ not unique:

$$\left.\begin{aligned} \vec{A}' &= \vec{A} + \vec{\nabla}\chi \\ U' &= U - \partial_t\chi \end{aligned}\right] \text{"gauge transformation"}$$

$$\Rightarrow \vec{B} = \operatorname{rot}\vec{A}', \quad \vec{E} = -\partial_t\vec{A}' - \vec{\nabla}U'$$

same fields

$\Rightarrow$ In classical physics, the motion depends only locally on the fields.

~~gauge: arbitrary ch~~

gauge potentials $\vec{A}, U$: only auxiliary!

Physics is gauge-invariant !
(same field → same result, independent of $\vec{A}, U$)

Lagrangian formulation:

$$L = \frac{m}{2}\dot{\vec{r}}^2 - qU(\vec{r},t) + q\dot{\vec{r}}\vec{A}(\vec{r},t)$$ "minimal coupling"

Check:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} = \frac{\partial L}{\partial \vec{r}}$$

$$\frac{d}{dt}\left[m\dot{\vec{r}} + q\vec{A}\right]_j = -q\partial_j U + q\dot{r}_\ell \partial_j A_\ell$$

$$\frac{d}{dt}A_j(\vec{r}(t),t) = \dot{r}_\ell \partial_\ell A_j + \partial_t A_j$$

$$\Rightarrow \quad m\ddot{r}_j = -q\partial_j U - q\partial_t A_j + q\underbrace{\left[\dot{r}_\ell \partial_j A_\ell - \dot{r}_\ell \partial_\ell A_j\right]}_{\left[\dot{\vec{r}}\times(\vec{\nabla}\times\vec{A})\right]_j}$$

$$m\ddot{\vec{r}} = q(\vec{E} + \vec{v}\times\vec{B}) \ \checkmark$$

Hamiltonian formulation:

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}} + q\vec{A}$$

$$\Rightarrow \quad \underbrace{m\dot{\vec{r}}}_{\text{"kinematic momentum"}} = \vec{p} - q\vec{A}$$

$\vec{p}$ → "dynamical momentum"

gauge change ⇒ $\vec{p}\,' = m\dot{\vec{r}} + q\vec{A}\,' = \vec{p} + q\vec{\nabla}\chi$

$$\Rightarrow \quad H = \dot{\vec r}\vec p - L = \ldots = \underbrace{\frac{1}{2m}(\vec p - q\vec A)^2}_{\text{kinetic energy}} + qU \quad (171)$$

thus: $\quad \vec p \mapsto \vec p - q\vec A \quad$ (& add qU)

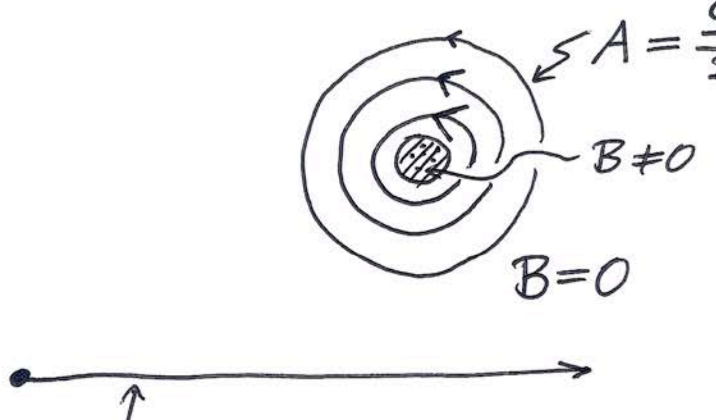
$$\Rightarrow \frac{d\vec r}{dt} = + \frac{\partial H}{\partial \vec p} = \frac{1}{m}(\vec p - q\vec A) = \vec v$$

$$\frac{dp_j}{dt} = - \frac{\partial H}{\partial r_j} = -q\partial_j U + \frac{q}{m}\dot r_l \partial_j A_l$$

(<u>not</u> $(\vec\nabla \times \vec A)_j$!)

$$\Rightarrow m\frac{d^2\vec r}{dt^2} = \ldots = F \checkmark$$

Example: magnetic flux through solenoid



, since

$$\oint \vec A d\vec s = 2\pi r A \underset{\text{Stokes}}{=} \int \mathrm{rot}\vec A\, d^2\vec f = \int \vec B\, d^2\vec f = \underset{\text{magnetic flux}}{\Phi}$$

trajectory unperturbed,

$\dot{\vec r} =$ const

but $\quad \vec p(t) = m\dot{\vec r} + q\vec A(\vec r(t))$

$$\Rightarrow \frac{d\vec p}{dt} \neq 0$$

Quantum mechanics:

$$\vec{p} \mapsto \hat{\vec{p}} = -i\hbar\vec{\nabla}$$

$$\hat{\vec{v}} = \frac{\hat{\vec{p}} - q\vec{A}}{m} \qquad \vec{\nabla} - i\frac{q}{\hbar}\vec{A} : \text{"gauge-invariant derivative"}$$

$$\hat{H} = \frac{1}{2m}(\hat{\vec{p}} - q\vec{A})^2 + qU$$

Gauge change ⇒

let $\quad i\hbar\,\partial_t \Psi = \hat{H}\Psi$

↷ with $\quad \Psi'(\vec{r},t) = \underbrace{e^{i\frac{q}{\hbar}\chi(\vec{r},t)}}_{\mathcal{U}(\vec{r},t)}\Psi(\vec{r},t) \equiv \mathcal{U}\Psi$

$\mathcal{U}(\vec{r},t)$ → unitary operator [diagonal in $\vec{r}$-basis]

we get:

$$(\hat{\vec{p}} - q\vec{A}')\mathcal{U} = \mathcal{U}(\hat{\vec{p}} + q\vec{\nabla}\chi - q(\vec{A} + \vec{\nabla}\chi))$$
$$= \mathcal{U}(\hat{\vec{p}} - q\vec{A})$$

⇒

$$i\hbar\,\partial_t\Psi' = \hat{H}'\Psi' \ \checkmark$$

[Note: $-q\partial_t\chi\Psi = \ldots \underbrace{-q\partial_t\chi}_{\text{from } U' = U - \partial_t\chi}\Psi$]

$$\boxed{\Psi' = \underset{\mathbb{C}}{\underset{\cap}{\mathcal{U}}}\,\Psi}$$

"U(1)" gauge transformation
(gauge ↳ local)

[→ extension to $\hat{\mathcal{U}} = \vec{r}$-dep. matrix in non-Abelian gauge fields]

Probability conservation:

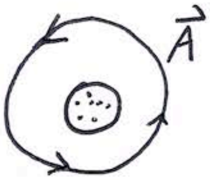
$$\partial_t|\Psi|^2 + \operatorname{div}\vec{j} = 0$$

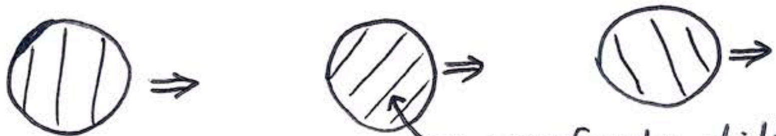
with $\quad \vec{j} = \mathrm{Re}\left[\Psi^* \hat{\vec{v}}\,\Psi\right]$

gauge-independent

(& used in Bohm's theory, for example)

Example:



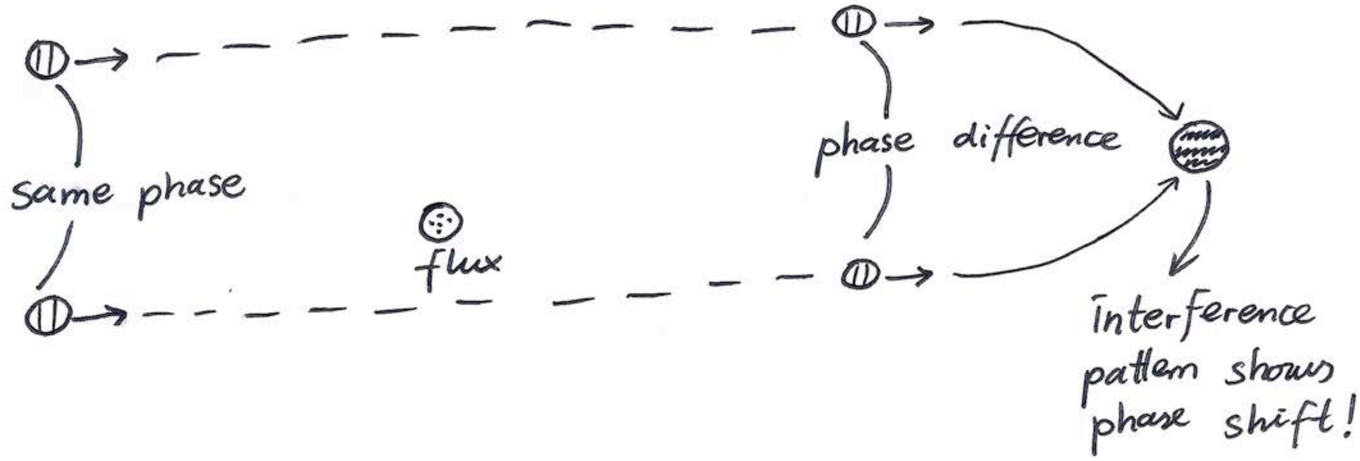


wavefronts tilted: $\langle \hat{\vec{p}} \rangle = m \langle \hat{\vec{v}} \rangle + q\vec{A}$

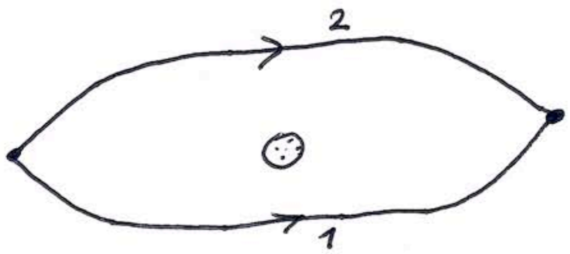
$$\frac{d\langle \hat{\vec{v}} \rangle}{dt} = 0 \,, \quad \text{but} \quad \frac{d\langle \hat{\vec{p}} \rangle}{dt} \neq 0$$

(like in classical case)

Double-slit experiment:



Path-integral approach:



$$\varphi = \frac{S[\vec{r}(\cdot)]}{\hbar}$$

action: $S = \int_{t_i}^{t_f} L \, dt$

Here: interference pattern

$$|\psi_1+\psi_2|^2 = |\psi_1|^2+|\psi_2|^2+\psi_1\psi_2^*+\psi_2\psi_1^*$$

here:

$$e^{+i\varphi_{AB}}$$

[semiclassical approx.: integrate along classical trajectory]

with

$$\underbrace{\varphi_{AB}}_{\text{Aharonov \& Bohm}} = \frac{1}{\hbar}\int_0^t \Big|_{\text{path 1}} q\,\vec{A}(\vec{r}(t))\,\underbrace{\dot{\vec{r}}\,dt}_{d\vec{r}} - \frac{1}{\hbar}\int\Big|_{\text{path 2}} \ldots$$

$$= \frac{1}{\hbar}q\oint \vec{A}\,d\vec{r} = \frac{q}{\hbar}\int \mathrm{rot}\,\vec{A}\,d^2\vec{f}$$

$$\boxed{\varphi_{AB} = \frac{q}{\hbar}\Phi}$$

or $\varphi_{AB} = 2\pi\frac{\Phi}{\Phi_0}$

with $\Phi_0 = \frac{h}{q}$ = flux quantum

$\approx 2\cdot 2{,}1\cdot 10^{-15}\,\mathrm{Tm}^2$ for $q = e^-$ charge

(in superconductivity, often define $\Phi_0^{(sc)} = \frac{h}{2q_e}$)

Ehrenfest, Siday '49

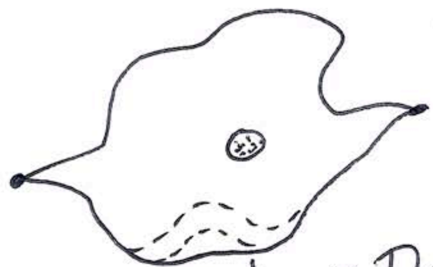
Aharonov, Bohm '59

Experiment: Chambers '60

$\Rightarrow$ In QM, observable effects depend nonlocally on the field or locally on the gauge potential!

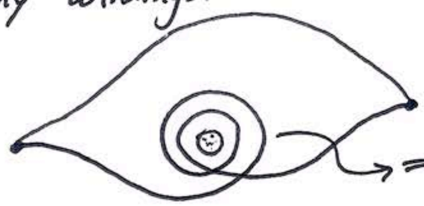
But: φ_{AB} is gauge-invariant ✓

Also: φ_{AB} is a <u>topological</u> phase



$\rightarrow$ Deformations of traj. don't change φ_{AB}

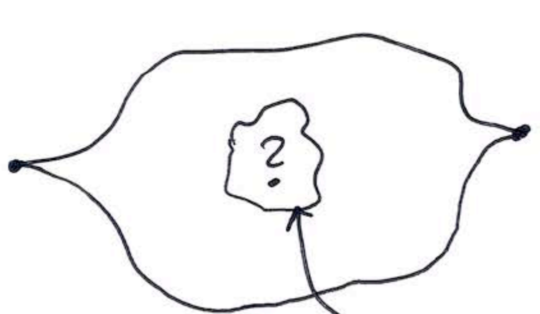
many windings:



$\rightarrow \Rightarrow$ additional $2\pi \cdot \frac{2\Phi}{\Phi_0}$

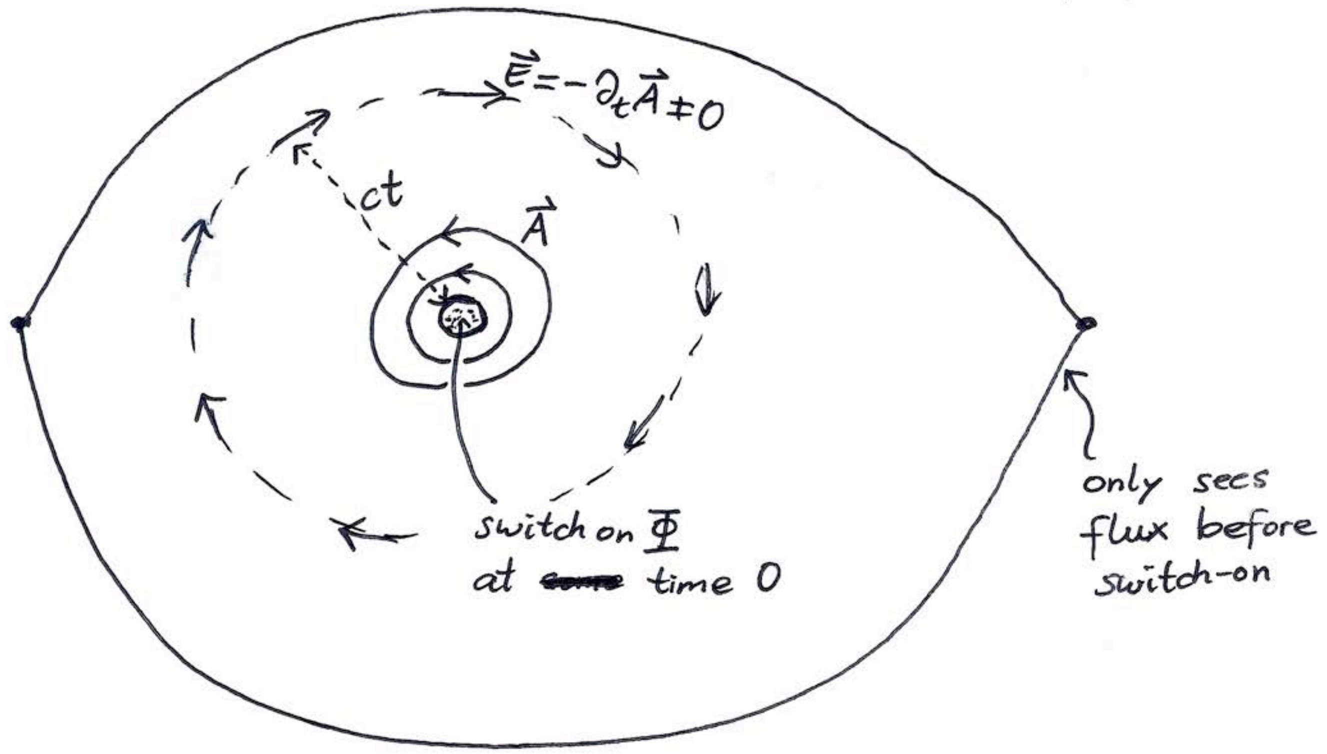
Quantization on multiply connected configuration space

$\Rightarrow$ possibility of topological phases!



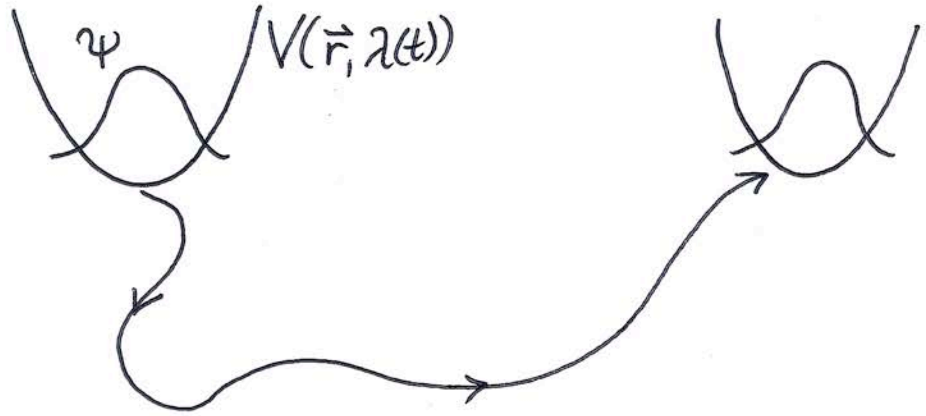
area inaccessible (for trajectories)

Nonlocal dependence on fields
$\Rightarrow$ signalling faster than light?
No!



7.2 "Berry phase"

Berry 1984
Pancharatnam '56



Adiabatic change of Hamiltonian

$$\hat{H} = \hat{H}(\lambda(t))$$

Solve SEQ:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}(\lambda(t)) |\psi(t)\rangle$$

Ansatz:

$$|\psi(t)\rangle = e^{i\varphi(t)} |\psi_0(\lambda(t))\rangle + \mathcal{O}(\dot{\lambda})$$

diagonalizes $\hat{H}(\lambda)$

$$\hat{H}(\lambda) |\psi_0(\lambda)\rangle = E_0(\lambda) |\psi_0(\lambda)\rangle$$

$\rightsquigarrow$ insert:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = -\hbar\dot{\varphi} |\psi(t)\rangle + e^{i\varphi} i\hbar \left[\frac{d}{d\lambda} |\psi_0(\lambda)\rangle\right] \dot{\lambda} \quad (+\ldots)$$

neglected terms

$$\stackrel{!}{=} \hat{H}(\lambda) |\psi(t)\rangle = E_0(\lambda(t)) |\psi(t)\rangle \quad (+\ldots)$$

$\langle \psi(t)|$ on both sides $\Rightarrow$

$$-\hbar\dot{\varphi} + i\hbar\dot{\lambda}\langle \psi_0(\lambda)|\frac{d}{d\lambda}|\psi_0(\lambda)\rangle = E_0(\lambda)$$

$$\Rightarrow \quad \dot{\varphi} = -\frac{1}{\hbar}E_0 + i\dot{\lambda}\langle\psi_0|\frac{d}{d\lambda}|\psi_0\rangle$$

$$\varphi(t) = \underbrace{-\frac{1}{\hbar}\int_0^t E_0(\lambda(t'))\,dt'}_{\text{"dynamical phase" (depends on time)}} + \underbrace{i\int_{\lambda(0)}^{\lambda(t)}\langle\psi_0|\frac{d}{d\lambda}|\psi_0\rangle \underbrace{d\lambda}_{\dot{\lambda}\,dt}}_{\text{geometric phase ("Berry phase"), only depends on trajectory!}}$$

Note: $\langle\psi_0|\frac{d}{d\lambda}|\psi_0\rangle \in i\mathbb{R}$

Proof: $\langle\psi_0|\psi_0\rangle \overset{!}{=} 1 \ \forall\lambda$

$\Rightarrow \langle\psi_0|\frac{d}{d\lambda}|\psi_0\rangle + \langle\frac{d}{d\lambda}\psi_0|\psi_0\rangle \overset{!}{=} 0$

$\Rightarrow \mathrm{Re}(\text{"}) = 0$

Note: $|\psi_0'(\lambda)\rangle = e^{i\tilde{\varphi}(\lambda)}|\psi_0(\lambda)\rangle$ (also solves SEQ for $\hat{H}(\lambda)$)

$$\Rightarrow \quad \varphi'(t) = \varphi(t) + \tilde{\varphi}(\lambda(0)) - \tilde{\varphi}(\lambda(t))$$

$\sim$ "gauge"-dependent

But for closed loops: <u>independent</u> of choice $\tilde{\varphi}(\lambda)$

Closed loops: Influence of dimension

$\lambda \in 1D$

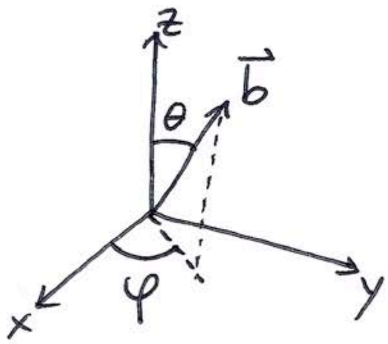
$\Rightarrow \varphi_B = 0$

$\lambda \in 2D$ or higher

$\Rightarrow \varphi_B \neq 0$ (maybe)

Example: <u>Spin</u> $(S=\tfrac{1}{2})$

$$\hat{H} = -\underbrace{\vec{b}(t)}_{\hat{=}\lambda}\cdot\hat{\vec{\sigma}}$$



$$|\psi\rangle = \begin{pmatrix}\psi_\uparrow \\ \psi_\downarrow\end{pmatrix}$$ quantized along z-axis

$|\vec{b}|$ "large" $\Rightarrow$ spin follows adiabatically!

e.g. $\langle\hat{\vec{\sigma}}\rangle \parallel \vec{b}$ (ground state)

$$\vec{b} = b\begin{pmatrix} s_\theta c_\varphi \\ s_\theta s_\varphi \\ c_\theta \end{pmatrix} \qquad \langle\hat{\vec{\sigma}}\rangle = \begin{pmatrix} \psi_\uparrow^*\psi_\downarrow + c.c. \\ -i(\psi_\uparrow^*\psi_\downarrow - c.c.) \\ |\psi_\uparrow|^2 - |\psi_\downarrow|^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2\,\mathrm{Re}\,\psi_\uparrow^*\psi_\downarrow \\ 2\,\mathrm{Im}\,\psi_\uparrow^*\psi_\downarrow \\ |\psi_\uparrow|^2 - |\psi_\downarrow|^2 \end{pmatrix}$$

$\rightsquigarrow$ have: $$\begin{pmatrix}\psi_\uparrow \\ \psi_\downarrow\end{pmatrix} = \begin{pmatrix} c_{\theta/2} \\ s_{\theta/2}\, e^{i\varphi}\end{pmatrix} \qquad [\Rightarrow \ldots \Rightarrow \langle\hat{\vec{\sigma}}\rangle \parallel \vec{b}\,]$$

now: $\vec{\lambda} = (\theta, \varphi)$

$$\Rightarrow \quad \left(\frac{d}{d\vec{\lambda}}\ldots\right)d\vec{\lambda} = \left(\frac{\partial}{\partial\theta}\ldots\right)d\theta + \left(\frac{\partial}{\partial\varphi}\ldots\right)d\varphi$$

$$\partial_\theta|\psi\rangle = \frac{1}{2}\begin{pmatrix} -s_{\theta/2} \\ c_{\theta/2}\, e^{i\varphi}\end{pmatrix}$$

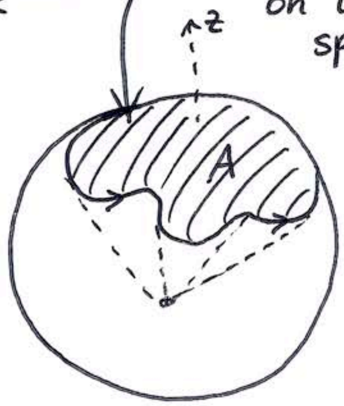
$$\partial_\varphi|\psi\rangle = \begin{pmatrix} 0 \\ i\, s_{\theta/2}\, e^{i\varphi}\end{pmatrix}$$

$$\Rightarrow \quad \langle\psi|\frac{d}{d\vec{\lambda}}|\psi\rangle\, d\vec{\lambda} = \frac{1}{2}\underbrace{\left(-c_{\theta/2}\, s_{\theta/2} + s_{\theta/2}\, c_{\theta/2}\right)}_{0} d\theta$$

$$+\; i\, s^2_{\theta/2}\, d\varphi$$

$$\Rightarrow \quad \varphi_{Berry} = -\oint \underbrace{s^2_{\theta/2}}_{\frac{1-c_\theta}{2}} d\varphi = -\frac{1}{2} A$$

area enclosed on unit sphere surface!



Geometry:

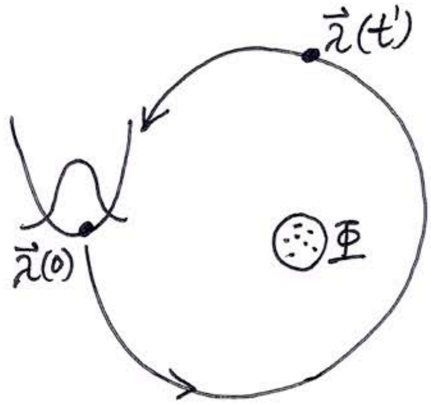
area

$$A = \int_A \sin\theta \, d\theta \, d\varphi$$

$$= \int_A d\cos\theta \, d\varphi$$

$$= \int 1 - \cos\theta \, d\varphi$$

AB-phase as Berry phase

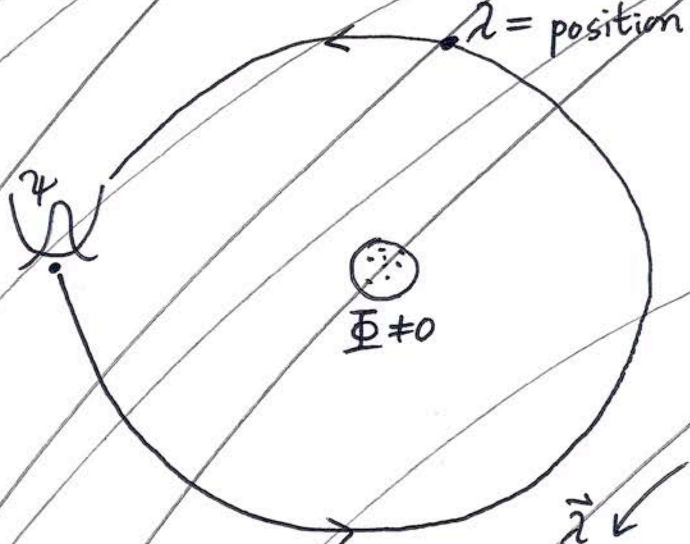


$$\Psi_0(\vec{r};\vec{\lambda}) = \exp\left[i\frac{q}{\hbar}\int_{\vec{\lambda}}^{\vec{r}}\vec{A}\,d\vec{s}\right]\Psi_0^{A=0}(\vec{r};\vec{\lambda})$$

$$\Rightarrow \ldots \Rightarrow \quad \langle\Psi_0|\frac{d}{d\vec{\lambda}}|\Psi_0\rangle = -i\frac{q}{\hbar}\vec{A}$$

$$\Rightarrow \ldots \Rightarrow \quad \varphi_B = +\frac{q}{\hbar}\oint\vec{A}(\vec{\lambda})\,d\vec{\lambda} = \frac{q}{\hbar}\Phi = \varphi_{AB}\checkmark$$

A.B. phase:



Let $|\psi_0(\vec{\lambda})\rangle = e^{i\frac{q}{\hbar}\int_0^{\vec{\lambda}} \vec{A} d\vec{s}} |\psi_0^{A=0}(\vec{\lambda})\rangle$ ← along path shown here

$$\Rightarrow \varphi_{Berry} = i \oint \langle \psi_0 | \tfrac{d}{d\vec{\lambda}} | \psi_0 \rangle d\vec{\lambda}$$

$$= -\frac{q}{\hbar} \oint \vec{A} d\vec{\lambda} = -\frac{q}{\hbar}\Phi = -\varphi_{AB} \checkmark$$

Define gauge potential from Berry phase:

let $\vec{A}_B \equiv i \langle \psi_0 | \frac{d}{d\vec{\lambda}} | \psi_0 \rangle$ ← dimensionality depends on dim. of $\vec{\lambda}$-parameter space!

$$\Rightarrow \varphi_B = \oint \vec{A}_B d\vec{\lambda}$$

$\Rightarrow$ Choice of $|\psi_0'(\lambda)\rangle = e^{i\tilde{\varphi}(\lambda)} |\psi_0(\lambda)\rangle$

is gauge-transformation:

$$\vec{A}_B' = \vec{A}_B - \frac{d}{d\vec{\lambda}} \tilde{\varphi}(\vec{\lambda})$$

7.3 Aharonov-Anandan phase

Arbitrary cyclic time-evolution

$$i\hbar\,\partial_t|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

~~Assume~~ If we have: $|\psi(T)\rangle = |\psi(0)\rangle \cdot e^{i\varphi(T)}$

($\Rightarrow$ return, up to a phase)

Then: $$|\psi(t)\rangle = \underbrace{|\psi_0(t)\rangle}_{\text{cyclic}}\, e^{i\varphi(t)}$$

$\Rightarrow \dots \Rightarrow$

$$\varphi(T) = -\frac{1}{\hbar}\int_0^T \langle\psi(t)|\hat{H}|\psi(t)\rangle\,dt + \varphi_{AA}$$

with $$\varphi_{AA} = i\oint \langle\psi_0(t)|\frac{\partial}{\partial t}|\psi_0(t)\rangle\,dt$$

8. Particle statistics

8.1 Fermions & bosons

Identical particles $\Rightarrow$

$$m_1 = m_2 = \dots$$

$$V(x_1, x_2, \dots) = V(x_2, x_1, \dots)$$

$\Rightarrow$ $\hat{H}$ invariant under $\underbrace{x_i \leftrightarrow x_j}_{\text{(elementary) permutation } \hat{P}_{ij}}$

$$[\hat{P}_{ij}, \hat{H}] = 0 \quad \forall i,j$$

$\Rightarrow$ Assume $\hat{P}_{ij}\psi = s\psi \quad \forall i,j$

at time 0 $\Rightarrow$ True for <u>all</u> times! *

Since $\hat{P}_{ij}^2 = \mathbb{1}$ (!) $\Rightarrow$

$$s^2 \overset{!}{=} 1 \Rightarrow s = \pm 1$$

$s = +1$: bosons

$s = -1$: fermions

* $\hat{P}_{ij}\hat{H}|\psi\rangle = \hat{H}\hat{P}_{ij}|\psi\rangle = s\,\hat{H}|\psi\rangle$

$$\Rightarrow \hat{P}_{ij}\,\partial_t|\psi\rangle = s\,\partial_t|\psi\rangle$$

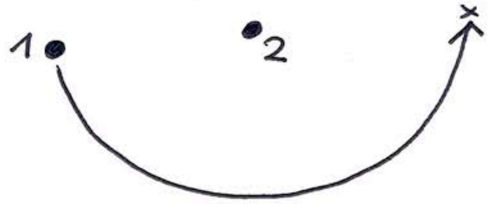
(anti-)symmetric always remains (anti-)symmetric!

Connection to spin → see relativistic QM

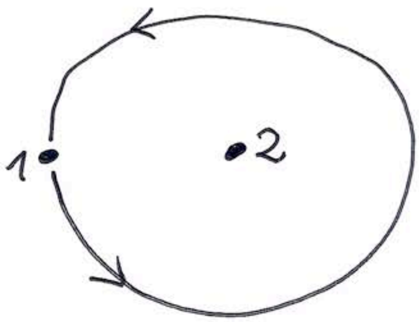
8.2 "Anyons"

In 2D: (may) have multiply connected configuration space

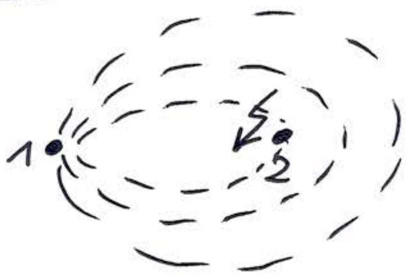
Particle exchange:



Exchange twice:

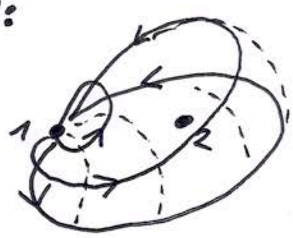


⇒ path <u>cannot</u> be contracted to 0



(e.g. prevented by interactions)

Contrast 3D:

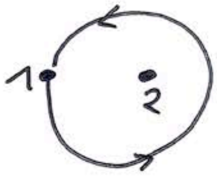


(lift out of plane)

⇒ in 2D :

can give phase $e^{i\theta}$

and θ is <u>arbitrary</u>, since

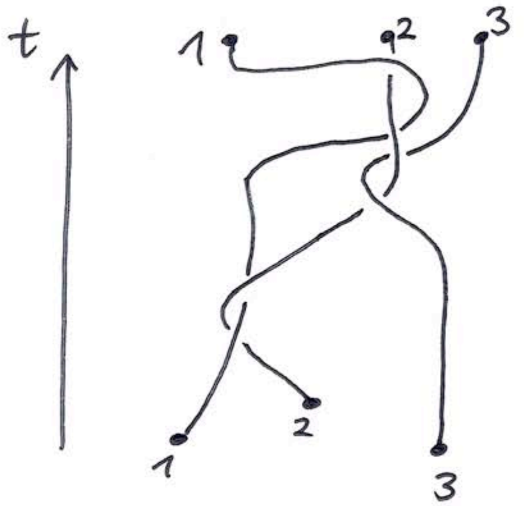


$e^{i2\theta}$ need <u>not</u> be 1

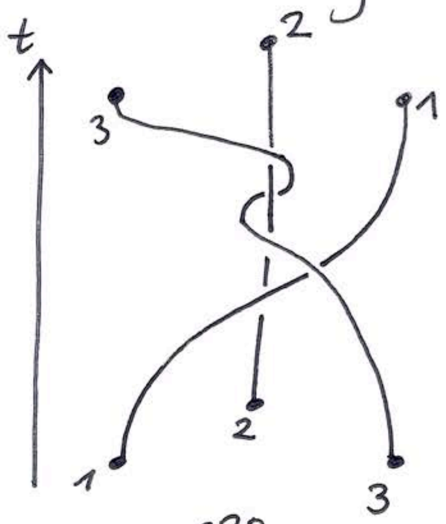
Note: 

gives $e^{-i2\theta}$

~~From~~ world lines: "braiding"



or



phase $e^{-i3\theta}$

formally : braid group = applying such interchanges in succession

= "fundamental group" of configuration space

identify all configurations that are only relabeled (particles are identical!)

$\Rightarrow$ Wave function Ψ will have different values depending on braid used to get to configuration! $\rightarrow$ multi-valued

[compare situation with fermions!]

Physical example:

Fractional Quantum Hall Effect (FQHE)

Skyrmions

1860 : Lord Kelvin

"atoms = knots in aether ?"

[⇒ inspired mathematical knot classification !]

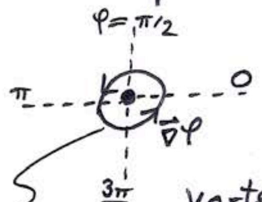
General idea:

Particles as localized topological excitations of fields

→ ensures particle conservation & discreteness

Example: Vortices in phase field $\varphi(\vec{r})$ (e.g.: superfluid / superconductor)

(2D)



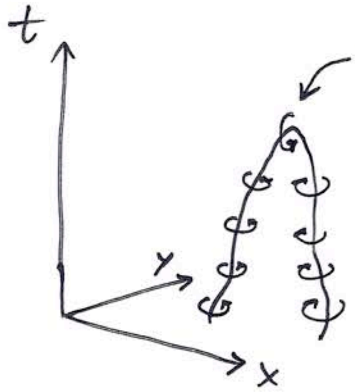
vortex anti-vortex

can annihilate (or be created pairwise)

Winding number:

$$\oint \vec{\nabla}\varphi \, d\vec{s} = \underset{\mathbb{Z}}{\underset{\cap}{n}} \cdot 2\pi$$

#vortices + #antiv. = conserved

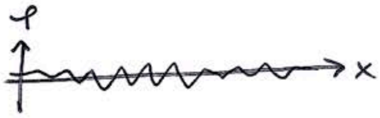


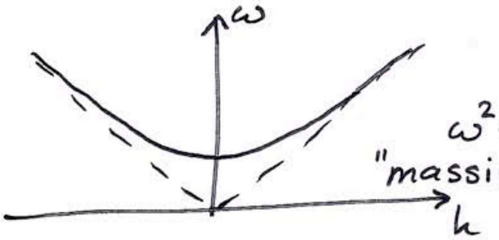
Simpler example:
(1D) Sine-Gordon field

$$\underbrace{\partial_t^2 \varphi - \partial_x^2 \varphi}_{\text{wave equation}} = \underbrace{-\sin\varphi}_{\text{nonlinearity} \rightsquigarrow \text{"interaction"}}$$

[all constants set to 1]

Small amplitudes: $\sin\varphi \approx \varphi \Rightarrow$



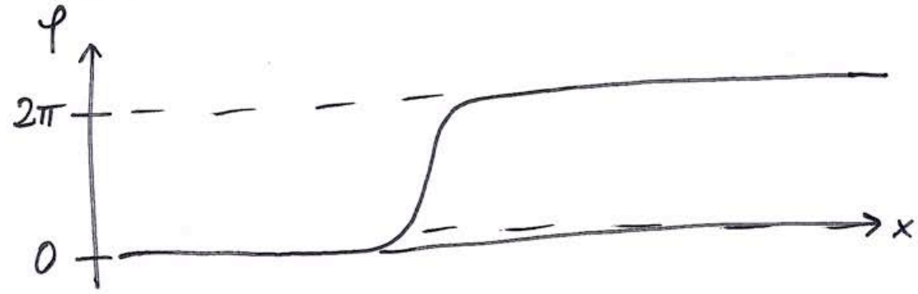


$\omega^2 = 1 + k^2$

"massive" linear field theory

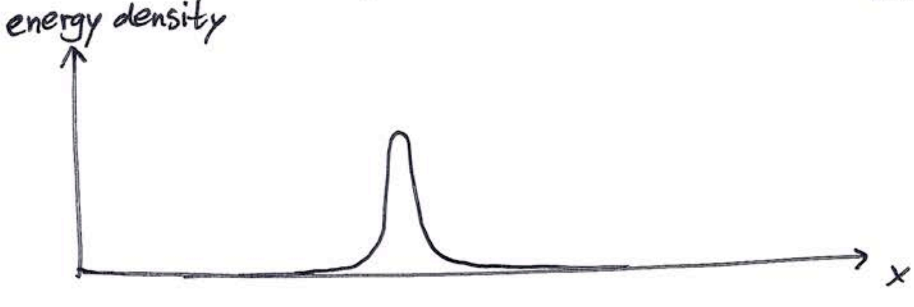
compare:
$E^2 = \hbar^2\omega^2 = m^2c^4 + \hbar^2 k^2 c^2$

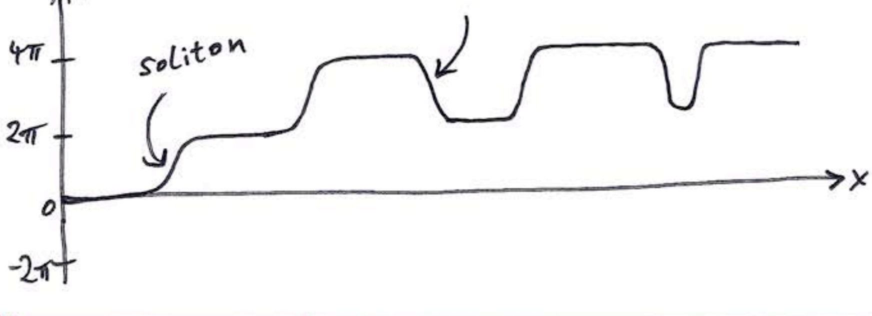
"soliton" solutions



$\Downarrow$

[Note: can move & Lorentz-contract]





Conserved "particle number":

$$\int_{-\infty}^{+\infty} \underbrace{\partial_x \varphi \, dx}_{=\varphi(\infty)-\varphi(-\infty)} = \underset{\mathbb{Z}}{\underset{\cap}{n}} \cdot 2\pi$$

(since $\varphi(\pm\infty) = n_{\pm} \cdot 2\pi$)

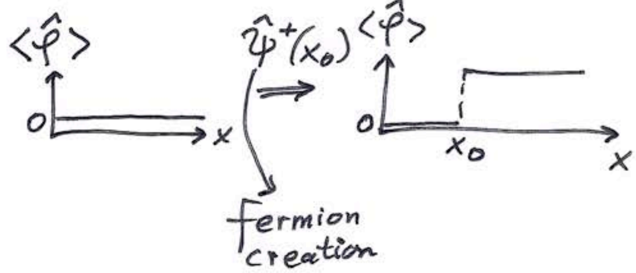
QM:
(Coleman et al., 70's)
(Skyrme 60's)

sine-Gordon model $\hat{=}$ massive Thirring model

sine-Gordon model → interacting <u>bosonic</u> field theory

massive Thirring model → Dirac <u>fermions</u> in 1 space D, with interactions (& mass)

solitons $\longleftrightarrow$ fermions



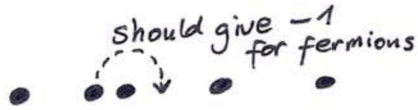
But note: 1D very special

e.g. hard-core bosons $\hat{=}$ free fermions

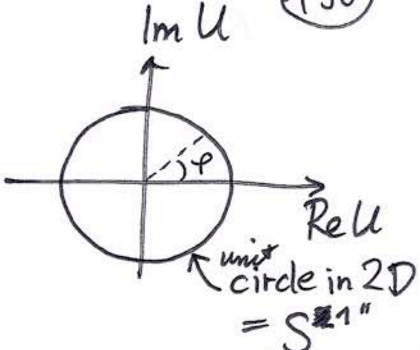
$$\hat{\psi}_F^{+}(x) \sim e^{i\pi \hat{N}(x)} \hat{\psi}_B^{+}(x)$$

$\hat{N}(x)$ → number of bosons at $x' < x$

"Jordan-Wigner transformation"

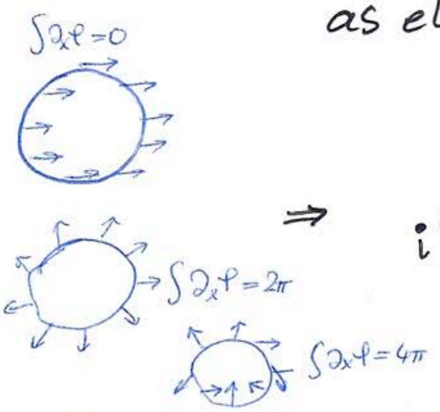


Skyrme: Consider field $U(x) = e^{i\varphi(x)}$ as elementary & demand



$$U(x \to \pm\infty) \to 1$$

$$\Rightarrow U : S^1 \to S^1$$



$$\Rightarrow \quad i\partial_x \varphi = U^+(x)\,\partial_x U(x) = -U(x)\,\partial_x U^+(x)$$

gives density for conserved particle number

$\Rightarrow$ Generalization to $\underline{\underline{3D}}$! ("Skyrme model")

Let $\hat{U}(\vec{r}) \in SU(2)$

$\hat{U}(\vec{r})$: $\underline{\text{classical}}$ matrix field

$SU(2)$: unitary matrices with $\det \hat{U} = 1$

$$\Rightarrow \hat{U}(\vec{r}) = e^{i\,\vec{\varphi}(\vec{r})\,\hat{\vec{\sigma}}/2} \rightarrow \text{rotation of spin } 1/2$$

$\vec{\varphi}(\vec{r})$: rotation axis & angle $|\vec{\varphi}|$

$\hat{\vec{\sigma}}$: Pauli matrices

~~other f~~ or, alternatively:

$$\hat{U}(\vec{r}) = \phi_0 \mathbb{1} + i\sum_{j=1}^{3} \phi_j \hat{\sigma}_j$$

$\phi_0, \phi_j \in \mathbb{R}$

Condition [for $\hat{U} \in SU(2)$]:

$$\sum_{\ell=0}^{3} \phi_\ell^2 = 1$$

$\rightarrow$ surface of sphere in 4D $\equiv$ "S^3"

$$\hat{U} : \mathbb{R}^3 \rightarrow \underbrace{S^3}_{3 \text{ parameters (e.g. } \vec{\varphi} \text{ or } \phi_1, \phi_2, \phi_3)}$$

Now demand:

$$\hat{U} \longrightarrow \mathbb{1} \quad \text{at} \quad |\vec{r}| \to \infty$$

$\rightsquigarrow$ effectively $\hat{U} : S^3 \longrightarrow S^3$

↳ identify $|\vec{r}| \to \infty$ as one point

Skyrme: writes down Lagrangian for $\hat{U}(\vec{r})$

$\rightsquigarrow$ minimize energy for static solutions

$\equiv$ "Skyrmions"

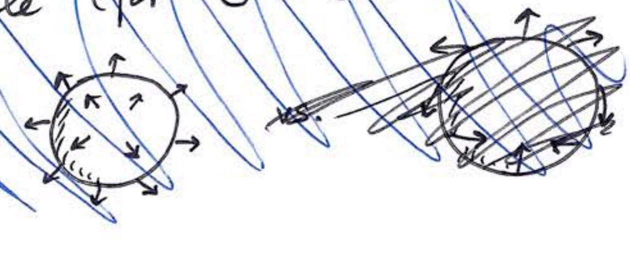
(3D analogues to 1D Sine-Gordon Solitons)

Topological invariant:

$$W = \frac{1}{24\pi^2} \varepsilon_{jkl} \int d^3\vec{r} \; \mathrm{tr}\left[\hat{U}(\partial_j \hat{U}^+)\hat{U}(\partial_k \hat{U}^+)\hat{U}(\partial_l \hat{U}^+)\right]$$

$$\in \mathbb{Z}$$

~~Example (for illustrated $S^2 \to S^2$)~~



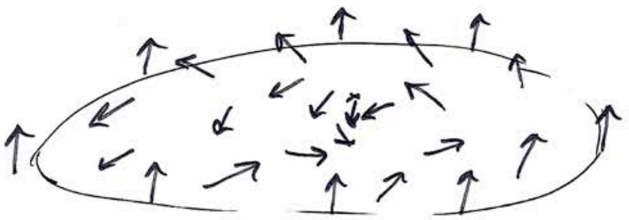
[Note: $S^2 \to S^2$ has $W = \frac{1}{8\pi} \int d^2\vec{r} \; \varepsilon^{lj} \; \vec{n} \cdot [\partial_l \vec{n} \times \partial_j \vec{n}]$

= number of times that $\vec{n}$ sweeps over sphere!]

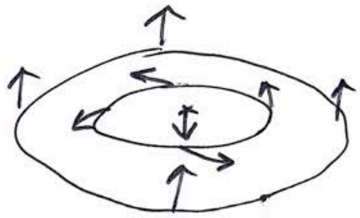
Example of Skyrmions:

Spins (in e.g. BEC or QHE or nanomagnets)

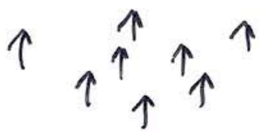
$$\langle \hat{\vec{\sigma}} \rangle \leftrightarrow \begin{pmatrix}\psi_\uparrow\\ \psi_\downarrow\end{pmatrix} = \hat{U}\begin{pmatrix}1\\0\end{pmatrix}$$



(pointing down in the middle)

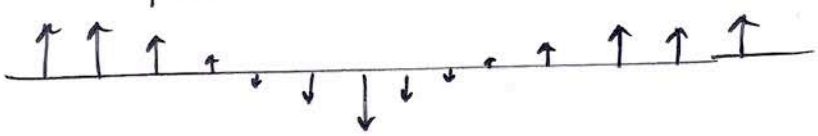


cannot be turned into

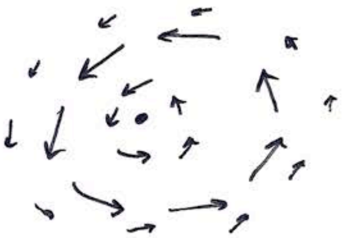


without $\langle \hat{\vec{\sigma}} \rangle \to 0$ at some point

from the side:



from above:



Topology of configuration space:

Configuration space of
Skyrme model is <u>not</u> simply
connected

"Fundamental group" $= \mathbb{Z}_2$

all closed loops

$\Downarrow$

can ~~assign~~
have factor
-1 associated
with exchange

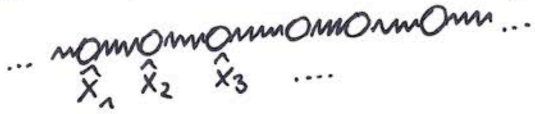
$\Downarrow$

fermions,
built from
nonlinear
bosonic
field theory

9. Quantum electrodynamics (QED)

9.1 Quantization

First QFT (1925):



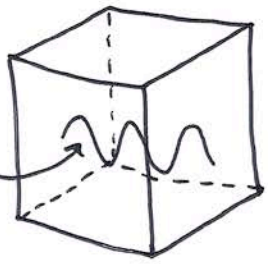
$\rightsquigarrow$ normal modes: $\hat{x}_j = \sum_k \phi_j^{(k)} \cdot \hat{u}_k$

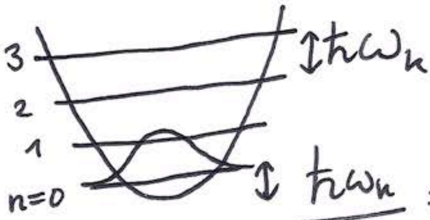
$\hat{u}_k$: harmonic oscillations at ω_k

QED: (1928 ff)

Field modes = harmonic oscillators

standing wave of e.m. field = field mode





$\frac{\hbar\omega_k}{2}$ = "zero-point energy"

$= \frac{1}{2}$(electric energy) $+ \frac{1}{2}$(magnetic energy)

$k = (\vec{k}, \sigma)$

$\vec{k}$: wave vector

σ: polarization (two transverse polarizations)

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^+ \hat{a}_k + \frac{1}{2}\right)$$

Use gauge with $U=0$ & $\operatorname{div}\vec{A}=0$

(here: use plane waves & periodic boundary conditions)

$$\Rightarrow \ldots \Rightarrow \hat{\vec{A}}(\vec{r},t) = \sum_k \vec{\mathcal{A}}_k \left(\hat{a}_k e^{i(\vec{k}\vec{r} - \omega_k t)} + h.c.\right)$$

with
$$\vec{\mathcal{A}}_k = \vec{\varepsilon}_k \cdot \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k}} \cdot \frac{1}{\sqrt{Vol}}$$

$\vec{\varepsilon}_k$: polarization vector

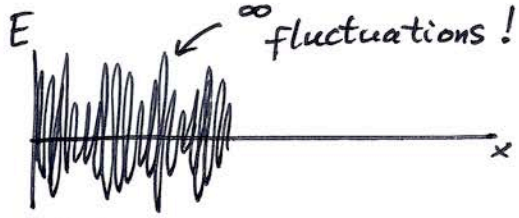
Vol: Volume of box ($\to \infty$ in the end)

$$\hat{\vec{E}}(\vec{r},t) = -\partial_t \hat{\vec{A}}$$
$$\hat{\vec{B}}(\vec{r},t) = \vec{\nabla} \times \hat{\vec{A}}$$

field fluctuations:

$$\langle 0|\hat{\vec{E}}^2(\vec{r},t)|0\rangle = \infty$$

$\hookrightarrow$ contributions from all wavelengths!



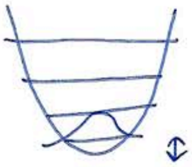
Smooth field: $\hat{\vec{E}}'(\vec{r},t) = \int f(\vec{r}-\vec{r}\,')\,\hat{\vec{E}}(\vec{r}\,',t)\,d\vec{r}\,'$

$\rightsquigarrow$ eliminate ~~cutoff~~ $\lambda < \lambda_c$ or $\omega > \omega_c$

$$\rightsquigarrow \langle 0|\hat{\vec{E}}'^2(\vec{r},t)|0\rangle \sim \omega_c^4 \xrightarrow[\omega_c\to\infty]{} \infty$$

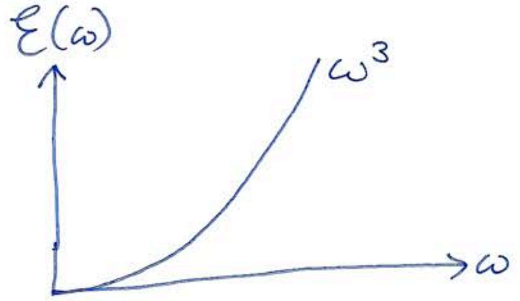
"ultraviolet divergence"

~~8~~ 9.2 The Vacuum: Casimir effect



$\updownarrow \frac{\hbar\omega}{2}$ = ground state energy

$$\Rightarrow \varepsilon = \frac{\text{Energy}}{\text{Vol}\cdot d\omega} = \frac{\hbar\omega}{2}\cdot\underbrace{D(\omega)}_{\substack{\text{density}\\ \text{of states}\\ \sim\omega^2 \text{ in 3D}}} \sim \omega^3$$



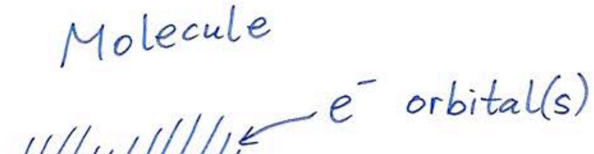
$$\hat{H} = \hbar\sum_k \omega_k\left(\hat{a}_k^{+}\hat{a}_k + \frac{1}{2}\right)$$

→ "just a constant offset" → remove?

OK, except when changing boundary cond's (→ changing ω_k!)

Is the ground state energy "physical"?

Yes! Example: Molecule



$E_0(\vec{R}_1, \vec{R}_2, \dots)$ = ground state energy of many-e^- system

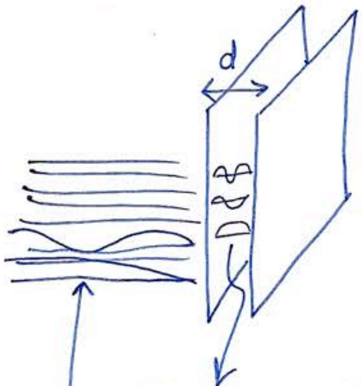
→ nuclear positions

$$\Rightarrow \vec{F}_j = -\frac{\partial E_0}{\partial \vec{R}_j} \quad \text{(+ Coulomb repulsion of nuclei)}$$

→ force on nucleus

→ real effect!

$\rightsquigarrow$ Change boundary cond's for e.m. vacuum!



field modes
between plates

modes outside: dense spectrum!

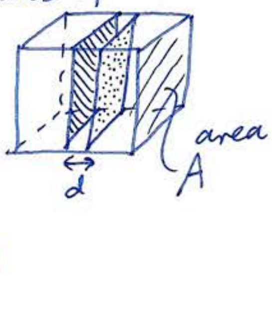
~~But~~ $$F = -\frac{\partial E_0}{\partial d}$$

but $E_0 = \sum_k' \frac{\hbar\omega_k}{2} = \infty \Rightarrow ?$

Artificial cutoff

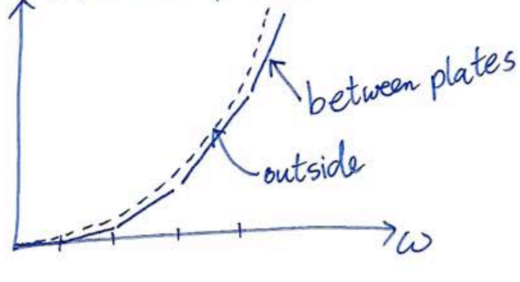
$$E_0 = \sum_k' \frac{\hbar\omega_k}{2} e^{-\frac{\omega_k}{\omega_c}}$$

modes of



cutoff
freq.
$\omega_c \to \infty$
in the end

$F = \lim_{\omega_c \to \infty} F(\omega_c)$ exists!



Casimir force:

$$F = -A \cdot \frac{\overbrace{\hbar c}^{\text{from dimensional considerations: } \hbar c \sim N m^4}}{d^4} \cdot \underbrace{\frac{\pi^2}{240}}_{\text{from detailed calculation}}$$

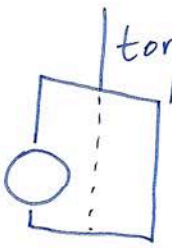
attractive

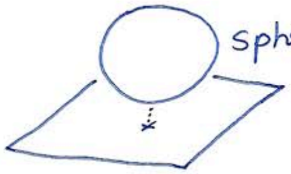
extensive ($\sim A$)

very strong for small distances!

Size: $\frac{F}{A} = 1.3 \cdot 10^{-3} \, \frac{N}{m^2} \cdot \frac{1}{(d/1\mu m)^4}$

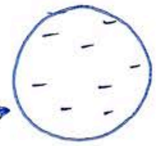
Expt. confirmed in '97 (Lamoreaux)





measure force (@ constant distance)

Casimir's idea: stabilize e^- against outward Coulomb repulsion?

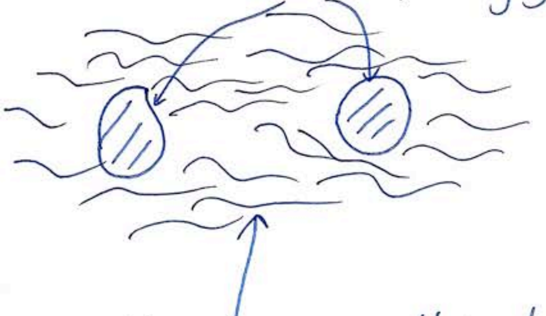


model of an e^-

No! Doesn't work: Casimir force on a spherical shell is <u>repulsive</u>.

Casimir forces in general:

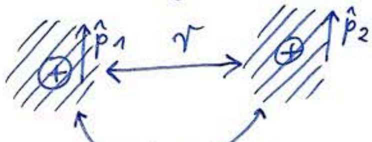
objects, changing boundary conditions



wave field, quantum or thermal noise (waves in liquids, em. waves, any field)

Casimir force vs. van-der Waals force

vdW:



interaction of dipoles:

$$\hat{H}_{int} \sim \frac{\hat{p}_1 \cdot \hat{p}_2}{r^3}$$

$$\Rightarrow \delta E^{(2)} \sim -\frac{1}{r^6} \equiv U_{vdW}(r)$$

2^{nd} order pert. thy (since $\langle \hat{p}_j \rangle \equiv 0$ in unperturbed ground state)

Casimir-Polder:

$$U_{CP}(r) \sim -\frac{C}{r^7}$$

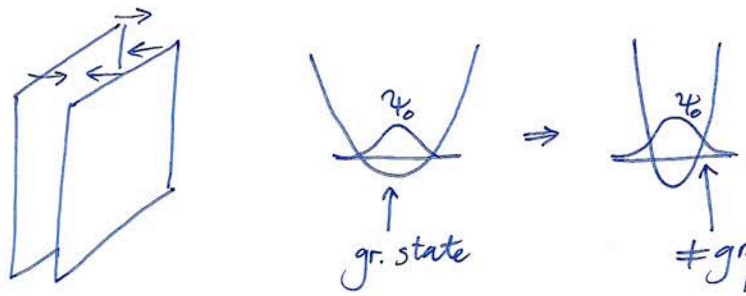
retardation!

relevant for $r \gg \lambda$

where λ: atomic transition wavelength

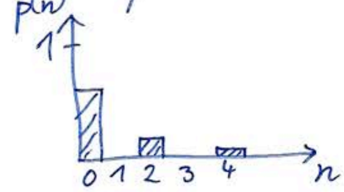
"Dynamical Casimir effect"

Sudden change $\Rightarrow$ photon production



gr. state

$\neq$ gr. state in new pot.: "squeezed state"!



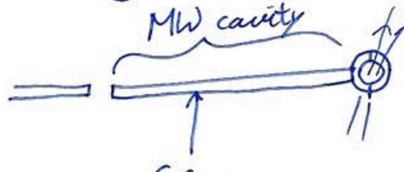
Challenge: <u>avoid</u> adiabatic regime $\dfrac{\dot{\omega}_k}{\omega_k} < \omega_k$

$\left(\omega \sim \frac{c}{L}\right)$

$\dot{L} < c \quad \Leftrightarrow$ (because then $|\psi(t)\rangle$ follows ground state)

Experiments: with microwaves, (Chalmers group)

e.g. use SQUID



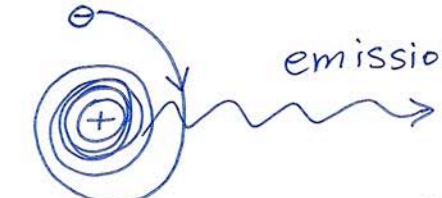
changes effective length of cavity!

s.c. metal strip

9.3 Stochastic Electrodynamics

(a physically motivated local hidden variable theory that didn't work...)

Classical instability of atom:

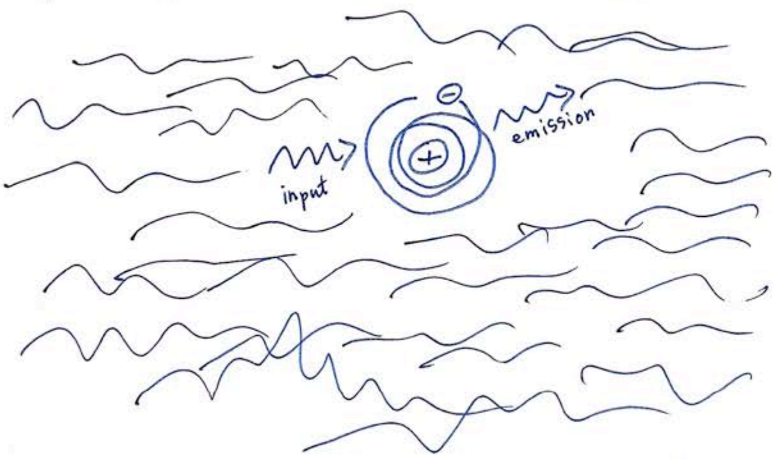


emission of radiation $\Rightarrow e^-$ spirals into nucleus

(in ~ nsec)

problem: damping = energy loss
but
no energy input!

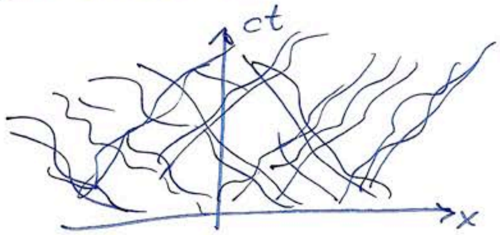
Idea: space filled with fluctuating em. field!



$\Rightarrow$ balance between dissipation & fluctuation!

~~compare: thermal equilibrium~~

Fundamental fluctuations $\rightarrow$ should be (statistically) Lorentz-invariant!



Characterize via correlator

$$\langle EE \rangle_\omega \equiv \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \langle E_x(t) E_x(0) \rangle$$

↑ at some point in space

Result: Lorentz-invariance $\Rightarrow$

$$\left(\text{Energy density } \mathcal{E}(\omega) \sim \right) \langle EE \rangle_\omega \overset{!}{=} \underbrace{\text{const}}_{\text{interpret as fundamental constant}} \cdot \underline{\underline{\omega^3}}$$

Use $\mathcal{D}(\omega) \sim \omega^2$ (in 3D) $\Rightarrow$ if $\mathcal{E}(\omega) \sim \omega^3$,

then: $\mathcal{E}(\omega) = \mathcal{D}(\omega) \cdot \frac{\text{En.}}{\text{mode}} \rightsquigarrow$

$$\Rightarrow \boxed{\frac{\text{Energy}}{\text{mode}} = \text{Const} \cdot \omega}$$

↓ purely from classical considerations, requiring Lorentz invariance!

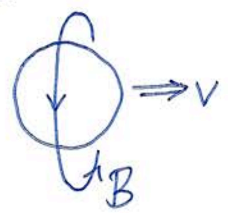
Experiment would show: $\text{Const} = \frac{\hbar}{2}$

$\rightsquigarrow$ this is how $\hbar$ enters SED!

Effects on an e^- ?

Side-remark: radiation reaction

charged sphere ⇒



accelerate ⇒

E generated via $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

⇓

opposes acceleration

⇓

"electromagnetic mass"

Result:

$$m_0 \ddot{x} = \underbrace{-\delta m\, \ddot{x}}_{\text{e.m. mass correction}} + \underbrace{\frac{q^2}{6\pi \varepsilon_0 c^3} \dddot{x}}_{\text{"radiation reaction" force}} + \dots$$

e.m. mass correction

$$\delta m = \frac{4}{3} \frac{E_{el}}{c^2}$$

$$E_{el} \sim \frac{q^2}{\varepsilon_0 R} \quad \leftarrow \text{radius}$$

"radiation reaction" force

note: $\delta m \xrightarrow[\text{for } R \to 0]{} \infty$ [⇒ classical e^- radius from $mc^2 \overset{!}{\sim} \frac{q^2}{\varepsilon_0 R} \Rightarrow R \sim 10^{-15}\,\text{m}$]

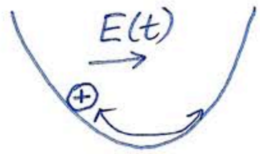
Note: $\ddot{x} = C \dddot{x}$ has solution $\dot{x}(t) = \dot{x}(0) + \ddot{x}(0)\, C \left(e^{t/C} - 1\right)$

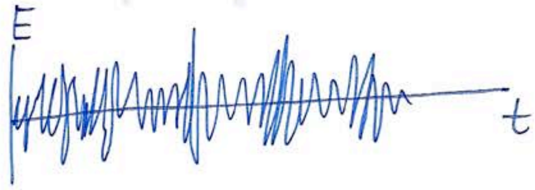
⚡ ⇒ exclude these solutions

apply external force ⇒ particle reacts "in advance"

Note: for e^-, $\frac{q^2}{6\pi \varepsilon_0 c^3 m_e} \sim 6 \cdot 10^{-24}\,\text{s}$

Charged oscillating particle subject to classical vacuum fluctuations





$$m\ddot{x} = -m\Omega^2 x + \underbrace{\frac{q^2}{6\pi\varepsilon_0 c^3}\dddot{x}}_{\approx -\underbrace{\left(\frac{q^2}{6\pi\varepsilon_0 c^3}\right)\Omega^2}_{\equiv m\Gamma}\dot{x}} + qE(t)$$

(e.g. already studied by Planck for thermal radiation!)

$\ddot{x} \approx -\Omega^2 x$

$m\Gamma \Rightarrow$ damping from rad. reaction!

$\Rightarrow$ solve in ω-space,

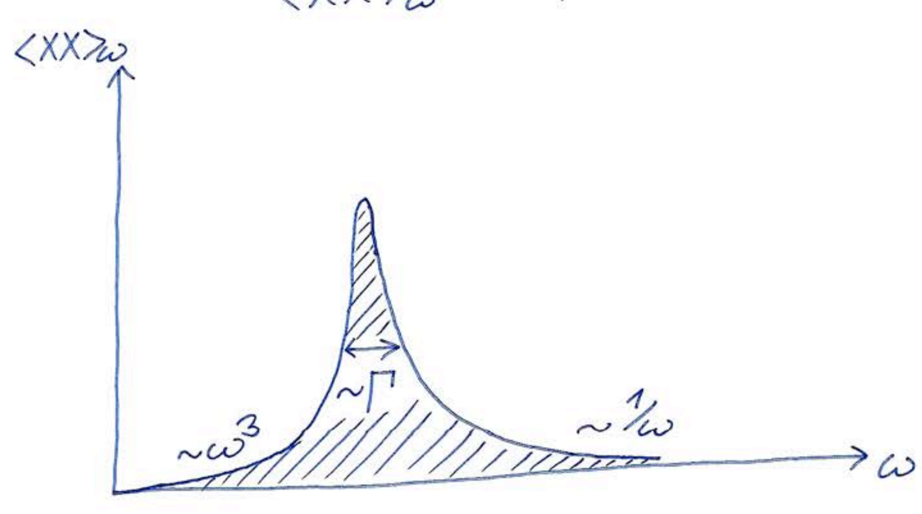
$$x(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} x(\omega)$$

$\Rightarrow$

$$x(\omega) = \underbrace{\frac{\frac{q}{m}}{\Omega^2 - \omega^2 - i\omega\Gamma}}_{\chi(\omega)} E(\omega)$$

$\Rightarrow$ Spectrum of fluctuations:

$$\langle XX\rangle_\omega = |\chi(\omega)|^2 \langle EE\rangle_\omega$$



Result:

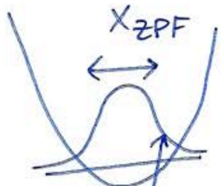
$$\langle x^2 \rangle = \int \frac{d\omega}{2\pi} \langle XX \rangle_\omega$$

$$= \ldots$$

$$= \underbrace{\frac{\hbar}{2m\Omega}}_{\text{QM ground-state width, } q^2 \text{ drops out!}} + \underbrace{\mathcal{O}(q^2)}_{\sim \ln\left(\frac{\omega_c}{\Omega}\right)}$$

||| QM ground-state width, q^2 drops out!

$\sim \ln\left(\frac{\omega_c}{\Omega}\right)$ Logarithmically divergent, but "small" $\sim q^2$ (like QED corrections)



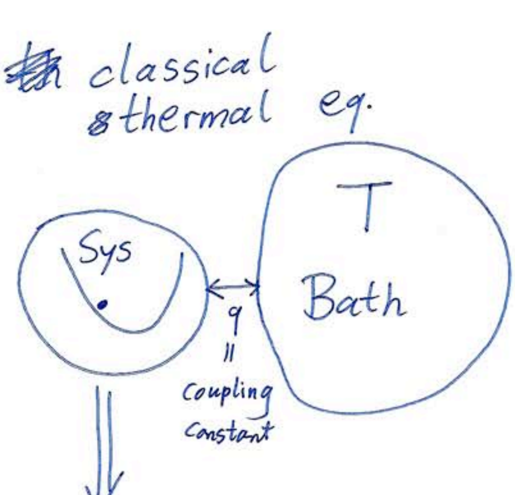
$\Rightarrow$ precisely matches QM predictions!

Gaussian distribution for x (due to Gaussian field E)

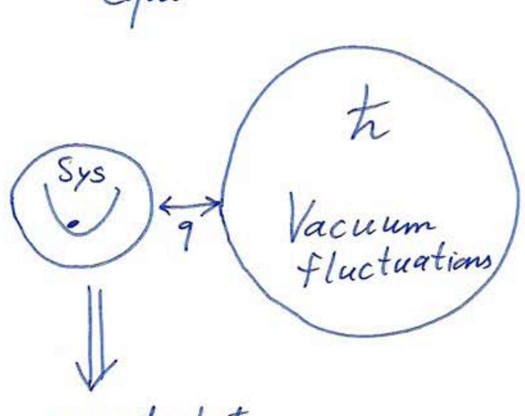
Compare:

~~cl~~ classical ~~&~~ thermal eq.

"zero-point equilibrium"



Boltzmann distr. (for $q \neq 0$, but $q \to 0$)



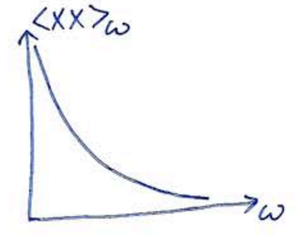
ground state (for $q \neq 0$, $q \to 0$)

Free particle:

$$m\ddot{x} = \underbrace{m\tau\dddot{x}}_{\text{(can be neglected for small } \omega\text{: } \tau(e^-)\sim 10^{-24}s)} + qE \quad \underbrace{(+q(v\times B))}_{\text{neglected for } v \ll c}$$

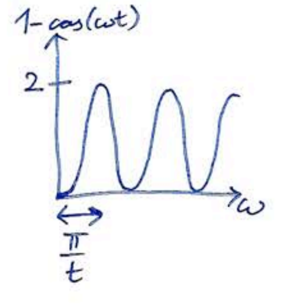
$$\Rightarrow \quad x(\omega) = \frac{\frac{q}{m}}{-\omega^2} E(\omega)$$

$$\Rightarrow \quad \langle XX \rangle_\omega = \frac{(q/m)^2}{\omega^4} \underbrace{\langle EE \rangle_\omega}_{\sim \omega^3} = \frac{1}{\omega} \cdot \text{Const}$$



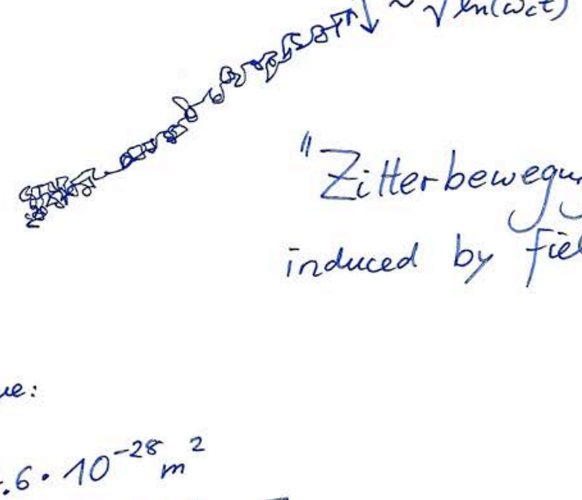
$$\left[\text{Note: } \langle EE \rangle_\omega = \underbrace{\frac{\hbar}{6\pi c^3 \varepsilon_0}}_{= \frac{\hbar \tau m}{q^2}} \omega^3 \right]$$

$$\langle (x(t) - x(0))^2 \rangle = \int \frac{d\omega}{2\pi} 2(1 - \cos(\omega t)) \langle XX \rangle_\omega$$



$$\approx \underbrace{\frac{1}{\pi} \text{Const} \cdot 2}_{\frac{2}{\pi} \frac{\hbar (q/m)^2}{6\pi c^3 \varepsilon_0}} \cdot \ln(\omega_c t) \quad \sim \ln(\omega_c t)$$

effective cutoff at $\sim \frac{1}{t}$



"Zitterbewegung" induced by field fluctuations

[Note: for e^-, we have:

$$\underbrace{\frac{2}{\pi} \frac{\hbar (q/m)^2}{6\pi c^3 \varepsilon_0}}_{\sim r \lambda_c} \approx 4.6 \cdot 10^{-28} m^2 = (2.15 \cdot 10^{-14} m)^2 \quad]$$

r ↳ classical e^- radius

Hydrogen atom: 

Estimate is OK (analogous to H.O.)

"Bach-of-the-envelope":

let $\Gamma = \Omega^2 \tau \quad (\Rightarrow m\ddot{x} = \ldots - \Gamma \dot{x})$

$$P_{in} = P_{out} \Rightarrow \underbrace{q\bar{E}v}_{\text{average power fed into motion}} \stackrel{!}{=} \underbrace{m\Gamma \langle \dot{x}^2 \rangle}_{\text{average power dissipated}}$$

$\bar{E} \sim \sqrt{\langle EE \rangle_{\omega=\Omega} \cdot \Gamma}$ → "bandwidth": filtering the noise during time Γ^{-1}

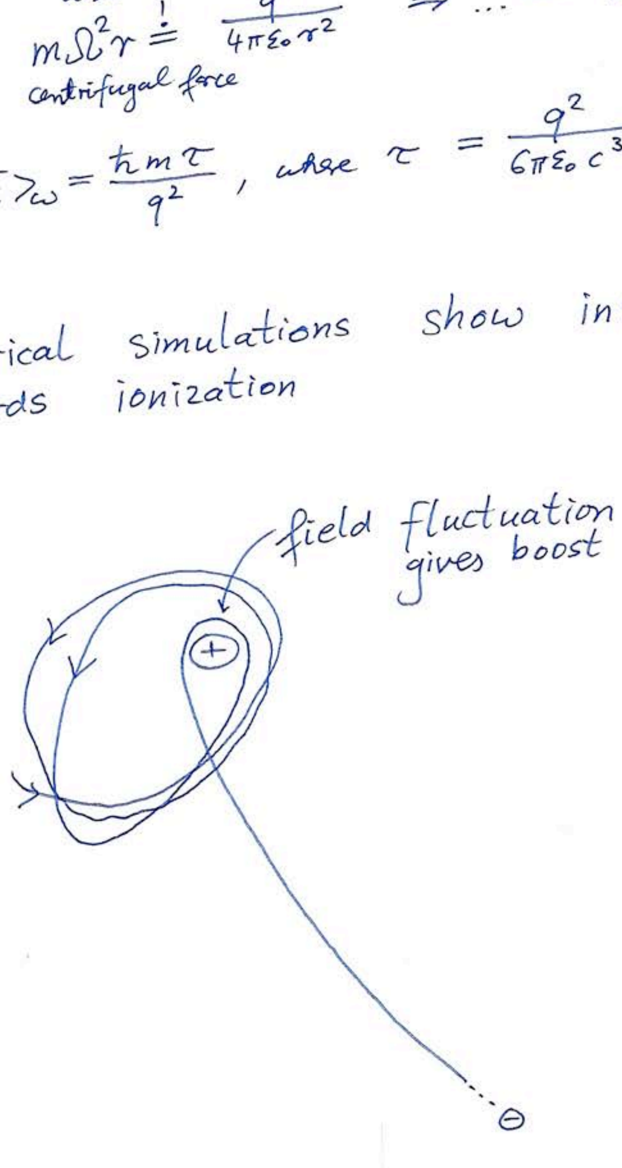
$\Rightarrow \ldots \Rightarrow r^2 \sim \frac{\hbar}{m\Omega}$

$(v \sim \Omega r)$

$\Rightarrow$ use $\underbrace{m\Omega^2 r}_{\text{centrifugal force}} \stackrel{!}{=} \frac{q^2}{4\pi\varepsilon_0 r^2} \Rightarrow \ldots \Rightarrow$ Bohr radius ✓

[Note: Use $\langle EE \rangle_\omega = \frac{\hbar m \tau}{q^2}$, where $\tau = \frac{q^2}{6\pi\varepsilon_0 c^3 m}$]

But: numerical simulations show instability towards ionization



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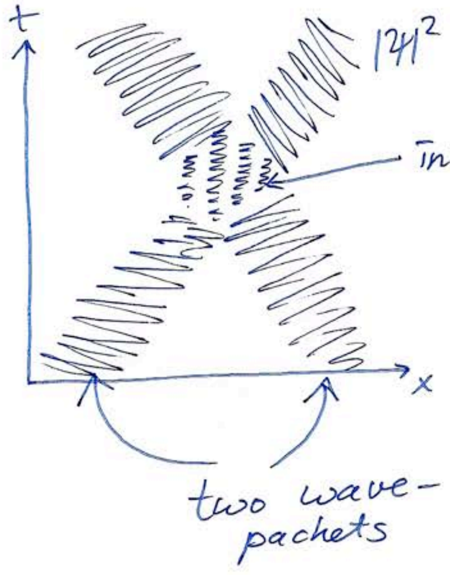
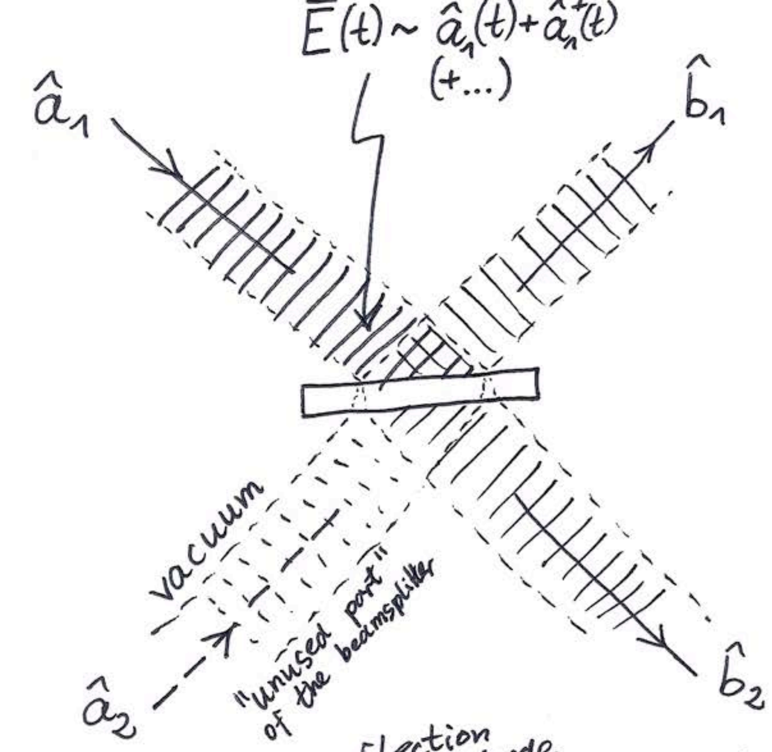
Even worse: interference in free space
cannot be explained!



interference pattern
$\Rightarrow$ trajectories must
be perturbed (quite strongly,
see Bohm or Nelson!)
SED field fluctuations
too weak for this!

9.4 The Vacuum in Quantum Optics

Vacuum at a beamsplitter



$$\hat{b}_1(t) = \underset{\text{reflection amplitude}}{r}\,\hat{a}_1(t) + \underset{\text{transmission amplitude}}{t}\,\hat{a}_2(t)$$

$$\hat{b}_2(t) = t\,\hat{a}_1(t) + r\,\hat{a}_2(t)$$

$[\hat{a}_i(t), \hat{a}_j^+(t')] = \delta_{ij}\,\delta(t-t')$ (independent input channels, & proper normalization)

$\langle \hat{a}_j^+(t)\hat{a}_j(t)\rangle$ = photons/sec (by normalization)

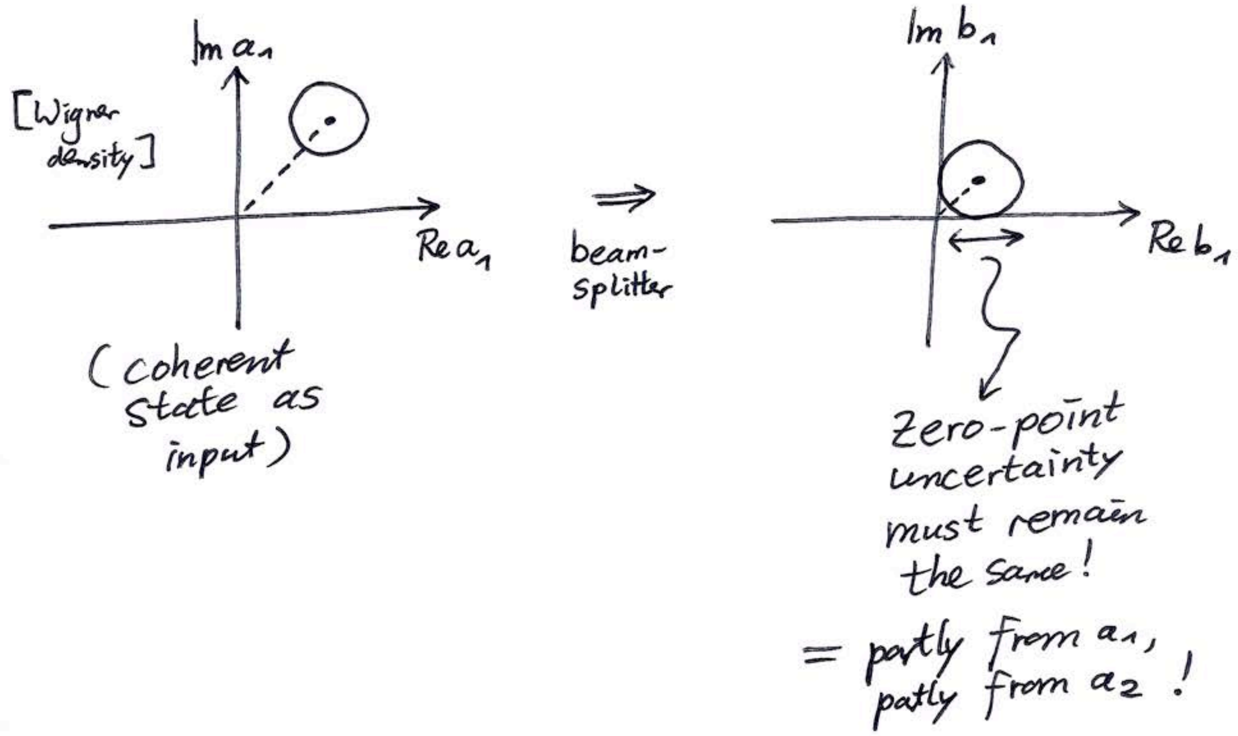
Must have energy conservation $\Rightarrow$

$$\underbrace{\langle \hat{b}_1^+\hat{b}_1\rangle + \langle \hat{b}_2^+\hat{b}_2\rangle}_{\parallel} \stackrel{!}{=} \langle \hat{a}_1^+\hat{a}_1\rangle + \langle \hat{a}_2^+\hat{a}_2\rangle$$

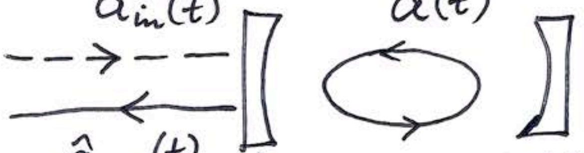
$$\underbrace{(|r|^2+|t|^2)}_{\stackrel{!}{=}1}(\langle \hat{a}_1^+\hat{a}_1\rangle + \langle \hat{a}_2^+\hat{a}_2\rangle) + \underbrace{(r^*t + t^*r)}_{\stackrel{!}{=}0}\underbrace{(\langle \hat{a}_1^+\hat{a}_2\rangle + \langle \hat{a}_2^+\hat{a}_1\rangle)}_{\text{could be } \neq 0 \text{ for interference setup}}$$

In QM, we need to keep $\hat{a}_2$ even if $\langle \hat{a}_2^+ \hat{a}_2 \rangle \equiv 0$!
Otherwise: commutation relations violated!

$$[\hat{b}_1(t), \hat{b}_1^+(t')] = |r|^2 \, [a_1(t), \hat{a}_1^+(t')] + |t|^2 \, [a_2(t), \hat{a}_2^+(t')] \overset{!}{=} \delta(t-t')$$



Vacuum entering a cavity



Heisenberg equations of motion
("input-output formalism")

vacuum enters here

$$\frac{d}{dt}\hat{a} = \left(-i\omega_{cav} - \frac{\kappa}{2}\right)\hat{a} \underbrace{- \sqrt{\kappa}\,\hat{a}_{in}(t)}$$

$$\hat{a}_{out}(t) = \hat{a}_{in}(t) + \sqrt{\kappa}\,\hat{a}$$

if this is forgotten,

then

$$[\hat{a}, \hat{a}^{+}] \to 0 \ \lightning$$

⇒ vacuum inside cavity is constantly replenished by incoming vac. noise
[compare SED picture for vac. acting on H.O.]

"The wave function of a photon"

Photon creation?

In mode k: $\hat{a}_k^+$ ✓

At position $\vec{r}$:

maybe $\hat{\Psi}^+(\vec{r}) = \frac{1}{\sqrt{V}} \sum_k \hat{a}_k^+ e^{-i\vec{k}\vec{r}}$?

Problem: incompatible with

$$\hat{\vec{E}}(\vec{r}) \sim \sum_k \sqrt{\omega_k}\, \vec{\varepsilon}_k (\hat{a}_k e^{i\vec{k}\vec{r}} + h.c.)$$

$\Rightarrow$ extra energy density

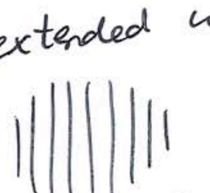
$$\delta\left[\frac{\varepsilon_0}{2}\langle \hat{\vec{E}}^2\rangle + \frac{1}{2\mu_0}\langle \hat{\vec{B}}^2\rangle\right]$$

change vs. Vac

is <u>not</u> $\sim \langle \hat{\Psi}^+(\vec{r})\hat{\Psi}(\vec{r})\rangle$

rather: <u>smeared</u> by $\sim\lambda$

$\Rightarrow$ unimportant for extended wave packets



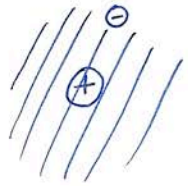
but it means: a single photon can never be localized to better than λ!

[Note: Here λ is analogous to Compton wavelength $\lambda_c \sim \frac{\hbar}{mc}$ of massive particles!]

8.5 Interaction with an atom

(e.g. H atom)

$$\hat{H}_{int} = -\underbrace{q\hat{\vec{r}}}_{\text{dipole moment}}\cdot\hat{\vec{E}}(0) \qquad \text{(in "dipole approximation", at atom } \vec{r}=0\text{)}$$



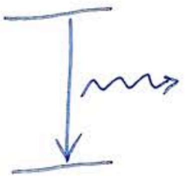
Perturbation theory $\Rightarrow$

Spontaneous emission rate: e.g. $|i\rangle = |2p_z\rangle$, $|f\rangle = |1s\rangle$

$$\Gamma_{f\leftarrow i} = \frac{1}{\hbar^2}\underbrace{|\langle f|\, q\hat{z}\, |i\rangle|^2}_{\text{dipole matrix element}} \langle\hat{E}\hat{E}\rangle_{\omega = E_f - E_i/\hbar}$$

where $\langle\hat{E}\hat{E}\rangle_\omega \equiv \langle\hat{E}_z\hat{E}_z\rangle_\omega$

$$= \int dt\, e^{i\omega t}\langle\hat{E}_z(t)\hat{E}_z(0)\rangle$$



~~Energy correction?~~

$$\cancel{\delta E_i^{(2)} = \sum_f \frac{|\langle f|\hat{H}_{int}|i\rangle|^2}{E_i - E_f} = \int\frac{d\omega}{2\pi}\frac{\langle\hat{E}\hat{E}\rangle_\omega}{-\hbar\omega}}$$

~~here:~~ $\cancel{|i\rangle = |i_{Atom}, Vac\rangle}$

$\cancel{|f\rangle = |f_{Atom}, 1_k\rangle}$ ~~1_k → one photon in some mode k~~

Purcell effect: "Shaping the vacuum"

Inside cavity: $\mathcal{D}(\omega)$ modified

$$\Rightarrow \quad \Gamma \sim \langle \hat{E}\hat{E} \rangle_{\omega=\Omega} \sim \omega \cdot \mathcal{D}(\omega)$$

is enhanced → at resonances
suppressed → between resonances

Heisenberg picture:

$$m\frac{d^2\hat{\vec{r}}}{dt^2} = -\vec{\nabla}V(\hat{\vec{r}}) + \underbrace{m\tau\frac{d^3\hat{\vec{r}}}{dt^3}}_{} + q\underbrace{\hat{\vec{E}}_{(0)}(\vec{r}=0,t)}_{}$$

damping, obtained from solving for E-field in presence of coupling:

vacuum field [here: taken in dipole approx.]

$\hat{\vec{E}}_{(0)}$ ← means: time-evolved in absence of coupling

$$\hat{\vec{E}}(\vec{r}=0,t) = \hat{\vec{E}}_{(0)}(\vec{r}=0,t) + \underbrace{\text{const.}\,\ddot{\hat{\vec{r}}}}_{\substack{\downarrow\\ \text{mass renormal.,}\\ \text{already included above!}}} + \underbrace{\text{const}'\cdot\dddot{\hat{\vec{r}}}}_{\substack{\downarrow\\ \text{radiation damping}}} \;(+\ldots)$$

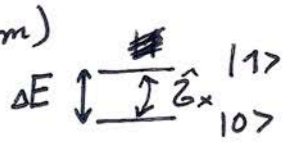
(→ like in SED, but with <u>operators</u>)

$\underbrace{\text{damping + vac. noise}}_{\substack{\downarrow\\ \text{both needed to keep commutators!}}}$

"Virtual" photon cloud

(e.g.: around an atom)

take atom as two-level atom



$$\hat{H}_{int} = \hat{\sigma}_x \sum_k g_k (\hat{a}_k + \hat{a}_k^+)$$

$$\hat{\sigma}_x = \hat{\sigma}^- + \underbrace{\hat{\sigma}^+}_{\text{excites atom}}$$

$\Rightarrow$ can have $\hat{\sigma}^+ \hat{a}_k^+$: increases energy by $\Delta E + \hbar\omega_k$

$\Rightarrow$ contributes correction

$$\frac{g_k}{\Delta E + \hbar\omega_k} |1\rangle \otimes \underbrace{|n_k = 1\rangle}_{\substack{\text{1 photon in mode } k \\ \text{(rest: vacuum)}}}$$

to ground state

$$|0\rangle \otimes |vac\rangle$$

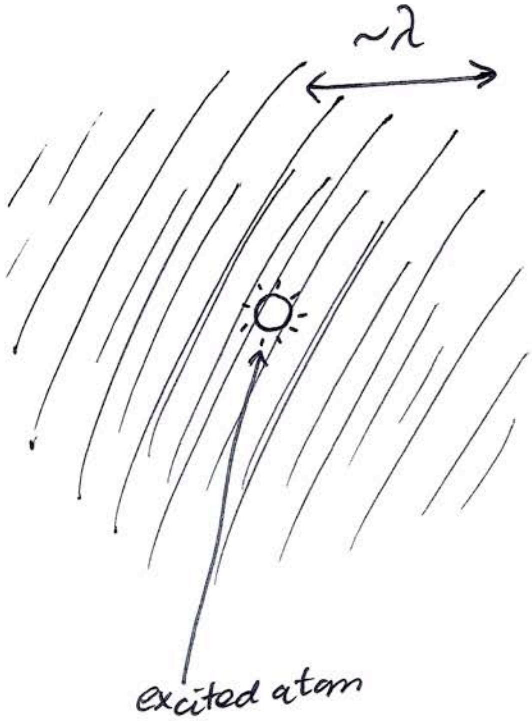
$\Rightarrow$ "Sudden" measurement would show (faster than $\hbar/\Delta E$)

~~sometimes~~
often:

(no photons)

sometimes:

Energy density of "photon cloud"

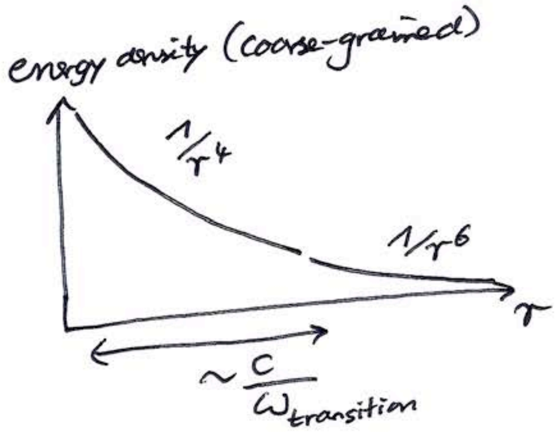


with

$\lambda =$ from transition ΔE

$\sim \frac{c}{\Delta E/\hbar}$

(?)



[see Passante, Compagno, Persico 1985]

8.6 Lamb shift

Energy correction? General formula: $\delta E_i^{(2)} = \sum_f' \frac{|\langle f|\hat{H}_{int}|i\rangle|^2}{E_i - E_f}$

Consider Atom level n, express via field spectrum

$\Rightarrow$

$$\delta E_n^{(2)} = \sum_{n'} |\langle n'|qz|n\rangle|^2 \int \frac{d\omega}{2\pi} \underbrace{\frac{\langle \hat{E}\hat{E}\rangle_\omega}{\varepsilon_n - \varepsilon_{n'} - \hbar\omega}}_{\text{pole at transition frequency}}$$

(& sum over different directions: x, y, z)

($\Rightarrow$ use principal value integral)

Observable: only transition frequencies

[note: $\hat{p}\cdot\hat{A}$ is more usual choice of gauge]

e.g. $\delta E_1^{(2)} - \delta E_0^{(2)}$

Problem: diverges! (at high ω)

- ~~Use~~ Don't use dipole approx. $\Rightarrow$ suppress short $\lambda \ll R_{Atom}$

- Use relativistic theory ($\rightarrow$ still: logarithmic divergence in cutoff!)
- Renormalize mass

$$\underbrace{m_{observed}}_{\text{fixed, from experiment}} = \underbrace{m_{eff}}_{\text{effective mass, calculated}}(\underbrace{m_0}_{\text{"bare mass", in Hamiltonian}}, \underbrace{\omega_c}_{\text{cutoff frequency}})$$

$\Rightarrow$ finite results!

e.g.: nonrelativistic calculation:

$$\delta m = \frac{4\alpha}{3\pi c^2} \int_0^{E_{max}} dE$$

$$m_{obs} \stackrel{!}{=} m_0 + \delta m(\omega_c)$$

Lamb shift:

Observation (1947) : $E(2S_{1/2}) \neq E(2^2P_{1/2})$, even though Dirac theory predicts them to be the same!

$$\delta E^{(2)}(\underbrace{2}_{n}\,{}^{\overset{\text{Spin }1/2}{2}}\underbrace{S}_{L=0}{}_{\underbrace{1/2}_{J=1/2}}) - \delta E^{(2)}(2\,{}^2\underbrace{P}_{L=1}{}_{\underbrace{1/2}_{J=1/2}}) \approx -1\,\mathrm{GHz}\cdot h$$

Heuristic explanation:

(Welton)

QED-"Zitterbewegung" (see SED discussion)

<u>smears</u> potential:

$$V_{eff}(\vec{r}) = \underbrace{\langle V(\vec{r}+\delta\vec{r})\rangle}_{\text{avg. over } \delta\vec{r},\ \cancel{\text{with}}} \approx V(\vec{r}) + \frac{1}{2}\underbrace{\langle \delta r_i \delta r_j\rangle}_{\delta_{ij}\frac{1}{3}\langle\delta\vec{r}^{\,2}\rangle}\partial_i\partial_j V + \ldots$$

$$\langle \delta\vec{r}^{\,2}\rangle \sim r_{e^-\text{radius}} \cdot \lambda_C \cdot \ln(\omega_c t)$$

↑ See above [SED section]

here: take cutoff $\omega_c \approx \frac{mc^2}{\hbar}$ (Compton wavelength)

$t \sim \frac{1}{\Omega}$ ← Bohr freq.

$$\Delta\frac{1}{r} = -4\pi\delta(\vec{r})$$

$$\langle\psi|\Delta V|\psi\rangle$$

$$= -4\pi\,|\psi(\vec{r}=0)|^2\cdot\frac{q^2}{4\pi\varepsilon_0}$$

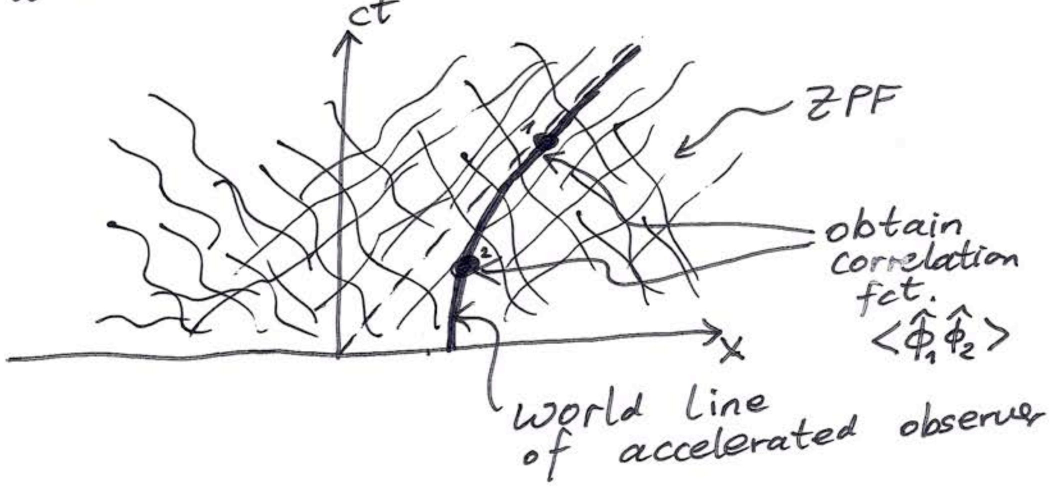
↳ only s-orbitals acquire correction!

9.7 Unruh-Davies Effect

(Unruh '76, Davies '75)

Zero-point spectrum is Lorentz-invariant
$\Rightarrow$ Ground state $|Vac\rangle$ is gr. state in <u>all</u> inertial frames!

But: What about accelerated observer?



Example: Scalar relativistic field theory

$$L = \frac{1}{8\pi}\int d\vec r \left[\frac{1}{c^2}(\partial_t\hat\phi)^2 - (\vec\nabla\phi)^2\right]$$

$$\Rightarrow \text{wave eq.} \quad (\vec\nabla^2 - c^{-2}\partial_t^2)\hat\Phi = 0$$

$\Rightarrow$ quantization:

$$\hat H = \sum_k \hbar\omega_k(\hat a_k^+\hat a_k + \tfrac{1}{2})$$

$$\hat\phi(\vec r,t) = \sum_k \sqrt{\frac{2\pi\hbar c^2}{Vol\,\omega_k}}\left[\hat a_k(t)e^{i\vec k\vec r} + h.c.\right]$$

$\Rightarrow$ Correlator:

$$\langle\hat\phi(\vec r,t)\hat\phi(0,0)\rangle = \sum_{k,k'}\frac{2\pi\hbar c^2}{Vol\sqrt{\omega_k\omega_{k'}}}\Big[\overbrace{\langle\hat a_k(t)\hat a_{k'}(0)\rangle e^{i\vec k\vec r} + \langle\hat a_k^+(t)\hat a_{k'}^+(0)\rangle e^{-i\vec k\vec r}}^{0} + \langle\hat a_k(t)\hat a_{k'}^+(0)\rangle e^{i\vec k\vec r} + \langle\hat a_k^+(t)\hat a_{k'}(0)\rangle e^{-i\vec k\vec r}\Big]$$

$$\langle\hat a_k^+(t)\hat a_{k'}(0)\rangle = e^{+i\omega_k t}\delta_{kk'}\,n_k$$

Vacuum state (gr. state):

$$\langle \hat{\phi}(\vec{r},t)\,\hat{\phi}(0,0)\rangle_0 = \frac{\hbar c}{\pi}\,\frac{1}{\vec{r}^{\,2}-c^2t^2}$$

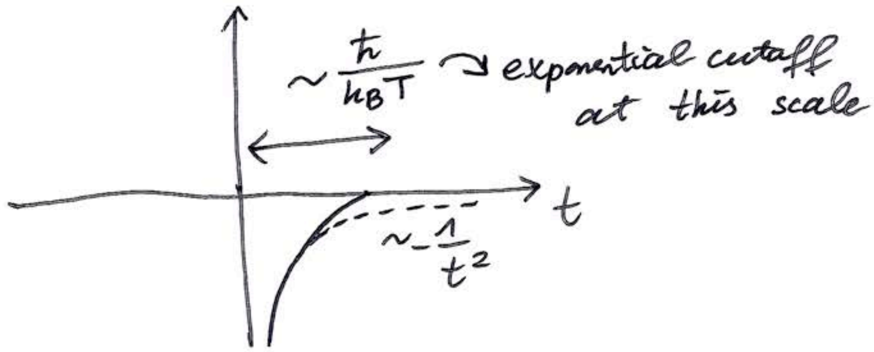
Finite temperature:

$$\langle \hat{a}_k^+ \hat{a}_k \rangle = \frac{1}{e^{\beta\hbar\omega_k}-1} = \text{Bose-Einstein distribution}$$

$$\langle \hat{a}_k \hat{a}_k^+ \rangle = 1 + \langle \hat{a}_k^+ \hat{a}_k \rangle \qquad \beta = \frac{1}{k_B T}$$

$$k \neq k' : \langle \hat{a}_k^+ \hat{a}_{k'} \rangle = 0$$

$$\Rightarrow \quad \langle \hat{\phi}(0,t)\,\hat{\phi}(0,0)\rangle_T = \ldots = -\frac{\hbar}{\pi c}\left(\frac{\pi}{\hbar\beta}\right)^2 \frac{1}{\sinh^2\left(\frac{\pi t}{\hbar\beta}\right)}$$



Observer with $\underbrace{\text{uniform acceleration}}$:

in its <u>own</u> frame

$$\Rightarrow \quad \frac{dv}{dt} = a\left(1-\left(\frac{v}{c}\right)^2\right)^{3/2}$$

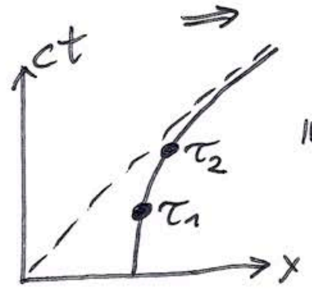
use $\dfrac{dt}{d\tau} = \cancel{\ldots} \dfrac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$ $\quad \tau$ = proper time

& solve $v(t) \Rightarrow \ldots \Rightarrow$

$$t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)$$

$$x(\tau) = \frac{c^2}{a}\left(\cosh\left(\frac{a\tau}{c}\right)-1\right)$$

$\Rightarrow$



"hyperbolic motion"

What is the correlator seen by the observer?

no transf. necessary: <u>scalar</u> field!

$$\langle \hat{\phi}(x(\tau_2), t(\tau_2))\, \hat{\phi}(x(\tau_1), t(\tau_1)) \rangle_0$$

$$= \frac{\hbar c}{\pi}\, \frac{1}{\delta x^2 - c^2 \delta t^2}$$

~~Use~~ $\cancel{\delta x^2 - c^2\delta}$

$$\delta t = t(\tau_2) - t(\tau_1)$$

We find:

$$\delta x^2 - c^2\delta t^2 = \ldots = -\frac{c^4}{a^2}\sinh^2\left(\frac{a(\tau_2-\tau_1)}{2c}\right)$$

only depends on $\tau_2 - \tau_1$!

$\Rightarrow$

$$\langle \hat{\Phi}(x(\tau_2), t(\tau_2))\, \hat{\Phi}(x(\tau_1), t(\tau_1)) \rangle_0$$
$$= \langle \hat{\Phi}(0, \tau_2)\, \hat{\Phi}(0, \tau_1) \rangle_T$$

with
$$k_B T = \frac{\hbar a}{2\pi c}$$

$\Rightarrow$ uniformly accelerated observer sees thermal fluctuations at this T! $\Rightarrow$ will sometimes excite observer! (= atom, for example)

Numbers: $a \sim 1g \Rightarrow T \sim 4 \cdot 10^{-20} K$

$a \sim 10^{12} \frac{m}{s^2} \Rightarrow T \sim 10^{-8} K$ $[6.5 \cdot 10^{-9} K]$ for $m = 10^{-25} kg$

(ion accelerated by $\frac{1 keV}{cm}$)

Note: "Hawking radiation": with a = grav. acceler. at horizon of black hole

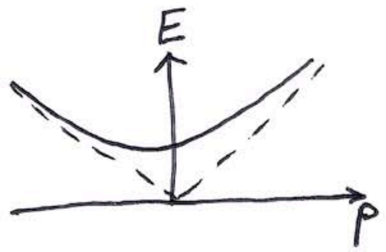
$\Rightarrow$
$$k_B T = \frac{\hbar c^3}{8\pi G M}$$

[Note: Schwarzschild radius $r_S = \frac{2GM}{c^2}$]

10. Dirac equation

$$E = \frac{p^2}{2m} \Leftrightarrow \hbar\omega = \frac{\hbar^2 k^2}{2m} \rightsquigarrow \ldots \text{ SEQ}$$

$$E = \sqrt{m^2c^4 + p^2c^2} \rightsquigarrow ?$$



1. attempt: $E^2 = \ldots$

$$\rightsquigarrow \quad -\hbar^2 \partial_t^2 \phi = \left(m^2c^4 - \hbar^2c^2 \vec{\nabla}^2\right)\phi$$

"Klein-Gordon eq."

[Schrödinger hadn't liked it: $E<0$ possible]

2. attempt: $\omega^2 = \ldots$ from <u>system</u> of 1^{st} order eqs (compare Maxwell eq.!)

e.g. 1D:

$$i\hbar\partial_t \begin{pmatrix}\psi_1\\ \psi_{-1}\end{pmatrix} = \underbrace{\left(mc^2 \hat{\sigma}_z + c\,\hat{p}\,\hat{\sigma}_x\right)}_{\hat{H}} \begin{pmatrix}\psi_1\\ \psi_{-1}\end{pmatrix}$$

[Note: energy eigenvalues of $\hat{H} = -\vec{b}\cdot\hat{\vec{\sigma}}$ are: $E = \pm|\vec{b}|$]

$$E^2 = \hat{H}^2 = m^2c^4 + mc^3\hat{p}\underbrace{\left(\hat{\sigma}_z\hat{\sigma}_x + \hat{\sigma}_x\hat{\sigma}_z\right)}_{=0} + c^2\hat{p}^2$$

also works in 2D, but for 3D: need $\underline{\underline{4}}$ anti-commuting matrices

$\Rightarrow$ 4x4 matrices!

$$i\hbar\partial_t\psi = \left(mc^2\hat{\beta} + c\,\hat{\vec{p}}\cdot\hat{\vec{\alpha}}\right)\psi$$

$\psi \hookrightarrow$ 4-component "spinor"

Dirac equation

$\hat{\beta}, \hat{\vec{\alpha}}$: many different choices possible $\hat{=}$ basis choices for ψ

with $\gamma^0 = \beta$
$\gamma^k = \gamma^0\alpha^k$

$$\Rightarrow \boxed{i\hbar\gamma^\mu\partial_\mu\psi - mc\psi = 0}$$

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

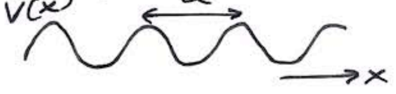
$$\gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} = \beta \qquad \gamma^1 = \begin{pmatrix} & \sigma_x \\ -\sigma_x & \end{pmatrix}$$

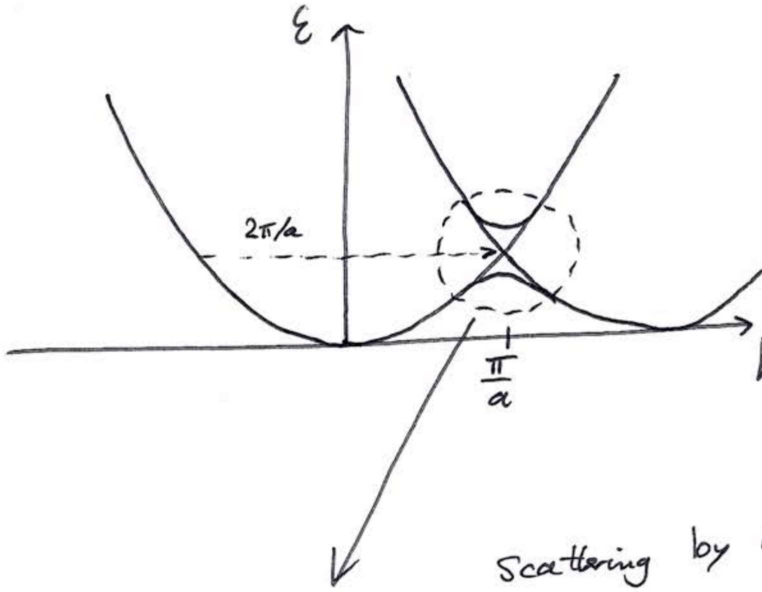
$$\gamma^2 = \begin{pmatrix} & \sigma_y \\ -\sigma_y & \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} & \sigma_z \\ -\sigma_z & \end{pmatrix}$$

$$\Rightarrow \alpha^1 = \begin{pmatrix} & \sigma_x \\ \sigma_x & \end{pmatrix} \text{ etc.}$$

Semiconductor analogue, e.g.: 1D periodic potential





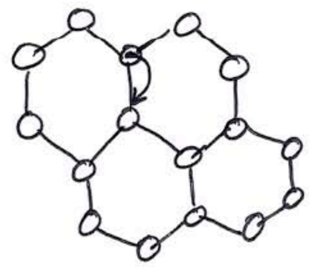
$$\hat{H} = \text{const} + \overbrace{V\left(q=\tfrac{2\pi}{a}\right)}^{\text{Scattering by reciprocal lattice vector}} \cdot \hat{\tau}_x + v \cdot \delta k \cdot \hat{\tau}_z$$

v: Speed at edge of Brillouin zone

$\delta k = k - \frac{\pi}{a}$

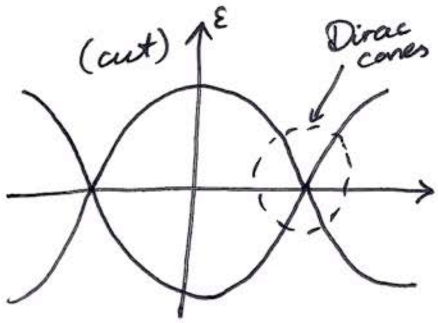
Also in graphene = 2D honeycomb lattice
($\rightarrow$ massless Dirac fermions)

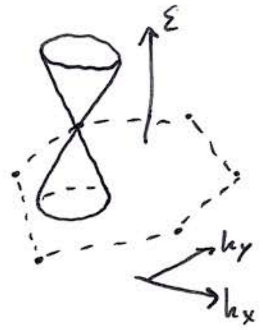
tight-binding model



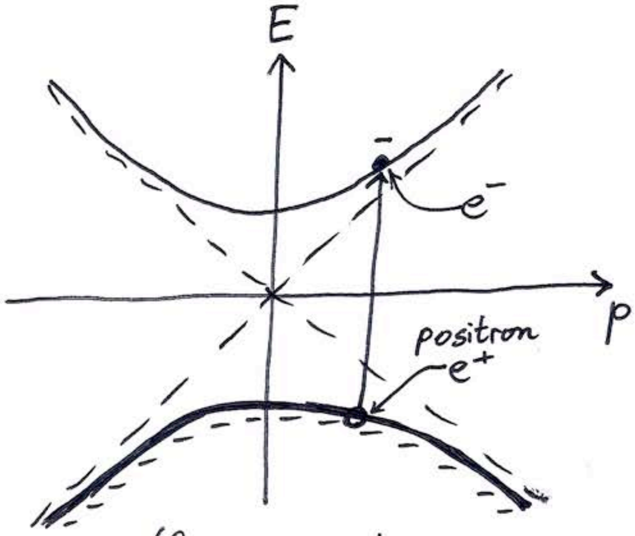
$\Rightarrow$

Bandstructure



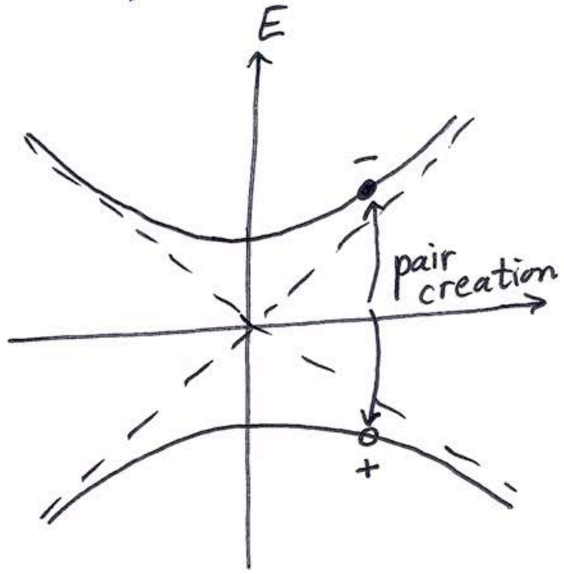


⇒ Automatically: Spin $\frac{1}{2}$ & particle/antiparticle (225)



(& compensating + background charge)

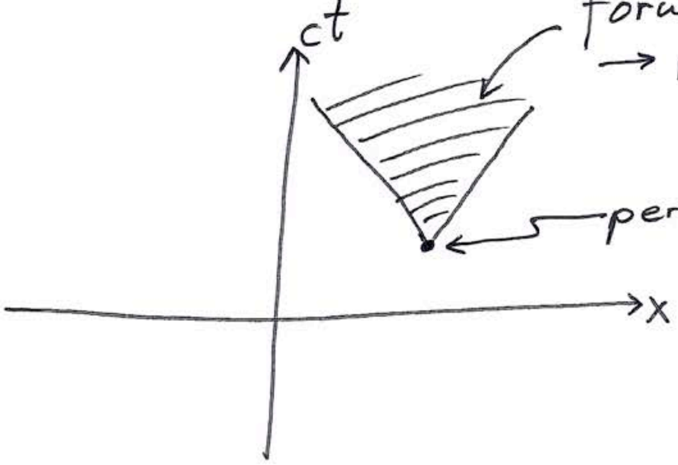
alternative point-of-view:



(& no background charge)

compare semiconductor!

Causality



perturbation:

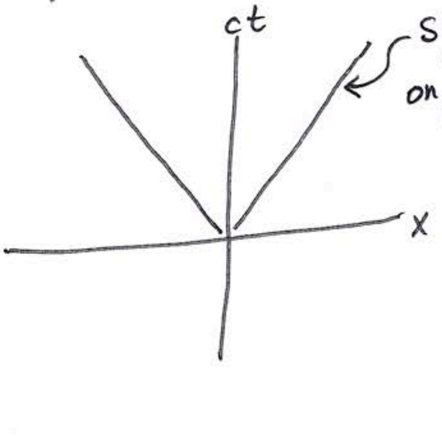
$$\hat{V} = \varepsilon \overset{\text{at pos. } x'}{\hat{\Phi}(x',t')}\ \overset{\text{acting at time } t'}{\delta(t-t')}$$

⇒ response, according to Kubo:

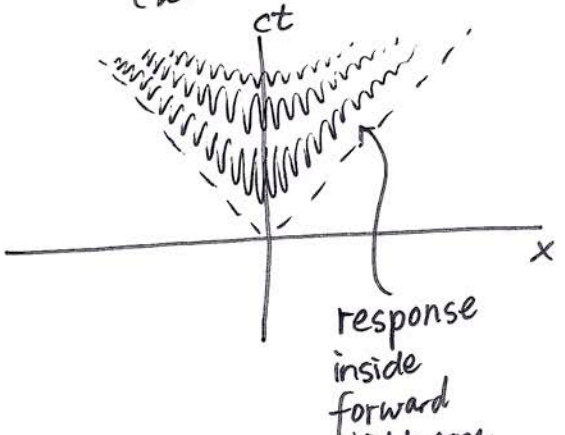
$$\langle \hat{\Phi}(x,t) \rangle = \varepsilon \frac{1}{i\hbar} \left\langle \left[\hat{\Phi}(x,t), \hat{\Phi}(x',t') \right] \right\rangle_0 \Theta(t)$$

Check for free bosonic relativistic fields:

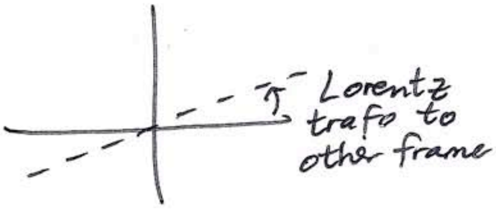
massless scalar $\hat{\phi}$



massive scalar $\hat{\phi}$ (Klein-Gordon)



Note: Need only check for equal times ("space-like separation") if we know the response fct. is Lorentz-invariant



$\rightsquigarrow \quad \langle [\hat{\phi}(x,t), \hat{\phi}(0,0)] \rangle = 0 \quad \text{for} \quad x^2 > c^2 t^2$ (227)

(Space like separation)

Fermions (e.g. Dirac eq.):

$$\langle \{ \hat{\psi}(x,t), \hat{\psi}^+(0,0) \} \rangle = 0 \quad \text{for} \quad x^2 > c^2 t^2$$

$\rightsquigarrow$ is compatible with bosonic case, since then Kubo response of density (= bosonic quantity) $\psi^+\psi$ is causal ✓

Note: Attempt of quantizing (spin-0) scalar $\hat{\phi}$ with anti-commutators $\Rightarrow$ violate causality ↯

$\langle \{ \hat{\phi}(x,t), \hat{\phi}(0,0) \} \rangle \neq 0$ for $x^2 > c^2 t^2$ ~~for $\hat{H}$ = const~~ $\omega_k = 0$

$\Rightarrow$ also observables quadr. in ϕ have problems!

Likewise (vice versa) for spin $1/2$ Dirac eq. $\Rightarrow$ ↯ from quantization with comm.

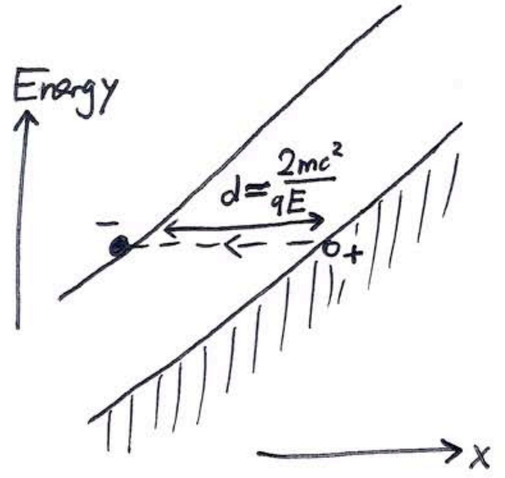
$\Rightarrow$ Examples of <u>spin-statistics theorem</u> (Pauli '40)

spin:

half-integer $\leftrightarrow$ fermions

integer $\leftrightarrow$ bosons

Pair creation in strong E-field



$$\Gamma \sim e^{-\mathrm{const} \cdot \frac{mc}{\hbar} \cdot d}$$

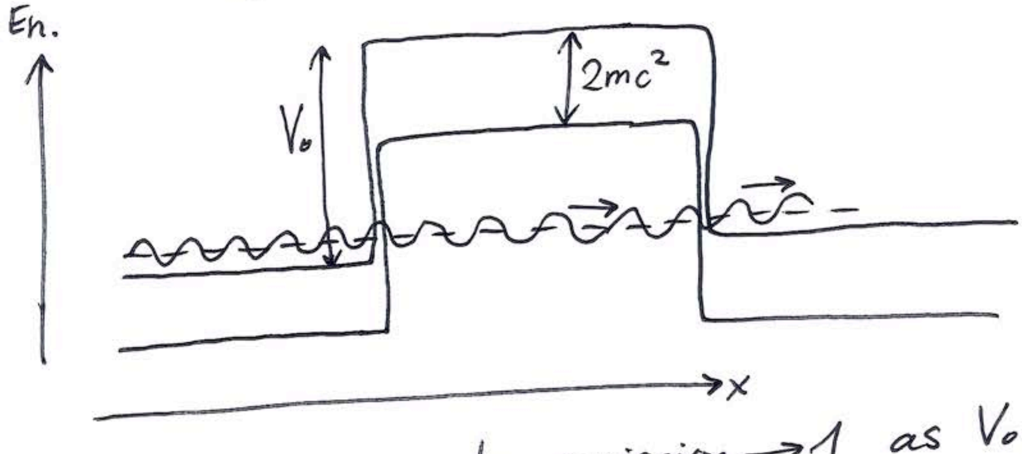
$$\sim e^{-\mathrm{const}\, \frac{m^2 c^3}{qE\hbar}}$$

$$\sim e^{-E/E_{cr}} \qquad E_{cr} \sim 10^{18} \frac{V}{m}$$

(!)

Sauter '31, Schwinger '51

"Klein paradox" '29 : Potential step

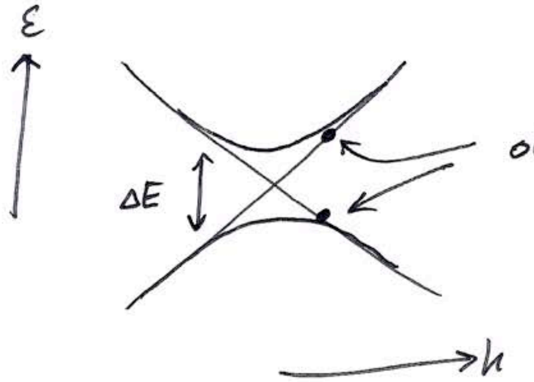


transmission $\to 1$ as $V_0 \to \infty$

(but: Coulomb forces tend to prevent formation of such a region!)

[$\to$ e.g. observed in graphene]

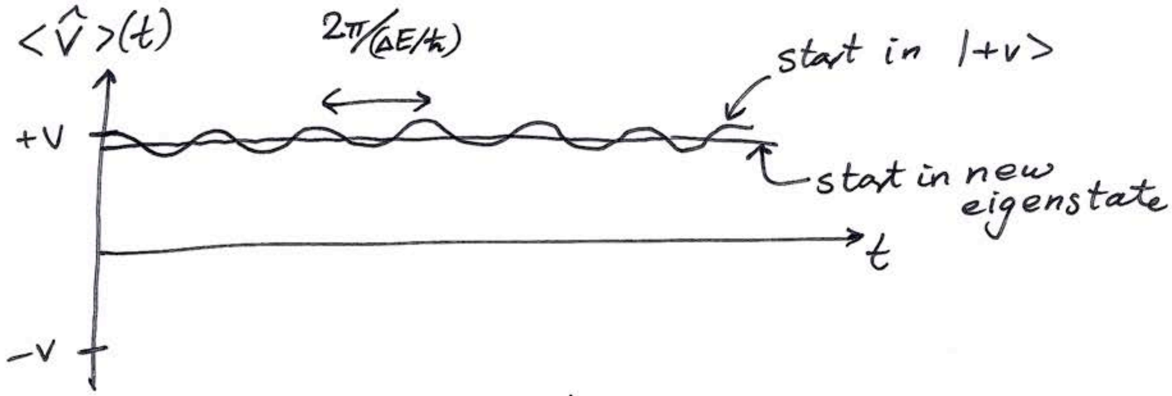
Zitterbewegung in 1D periodic potential



old states: eigenstates of velocity $|\pm v\rangle$

$$\Rightarrow \frac{d\hat{r}}{dt} = v\,\hat{e}_z$$

new states: superpositions!



Physical picture:
Repeated Bragg scattering!

"relativistic Zitterbewegung"

$$\hat{H} = mc^2\hat{\beta} + c\hat{\vec{p}}\cdot\hat{\vec{\alpha}}$$

$$\Rightarrow \quad \frac{d\hat{\vec{r}}}{dt} = \frac{1}{i\hbar}[\hat{\vec{r}}, \hat{H}] = c\cancel{\beta}\underbrace{\hat{\vec{\alpha}}}_{\text{eigenvalues } \pm 1}$$

$$\frac{d\hat{\vec{\alpha}}}{dt} = \frac{1}{i\hbar}[\hat{\vec{\alpha}}, \hat{H}] = 2\frac{i}{\hbar}(\hat{\vec{p}} - \hat{\vec{\alpha}}\hat{H})$$

$$\Rightarrow \quad \hat{\vec{\alpha}}(t) = (\hat{\vec{\alpha}}(0) - c\hat{\vec{p}}\,\hat{H}^{-1})e^{-2\frac{i}{\hbar}\hat{H}t} + c\hat{\vec{p}}\,\hat{H}^{-1}$$

oscillations in $\langle\hat{\vec{\alpha}}(t)\rangle$, if state contains $E>0$ & $E<0$

freq. $\sim \frac{2mc^2}{\hbar} \simeq 10^{21}\,\mathrm{Hz}$

Regularization

Exclude small λ/high ω
$\Rightarrow$ cutoff ω_c
[or otherwise keep results finite]

Renormalization

Fix "bare" parameters
by comparison with experiment

$$A_{observed} \stackrel{!}{=} A_{calculated}(\underbrace{\lambda_1, \lambda_2, \ldots, \lambda_N}_{\text{bare parameters}}, \omega_c)$$

In QED: renormalize

m
$q_e \rightsquigarrow$ vacuum polarization!
$\varepsilon_0 \rightsquigarrow$
ψ

[$\hbar$, c fixed]

Vacuum polarization