Lecture 14

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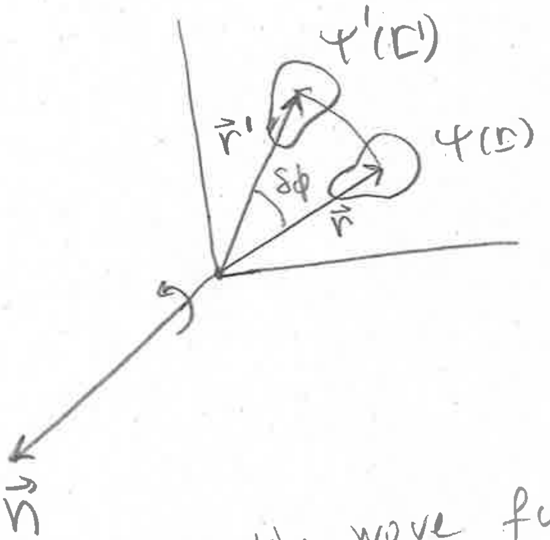
— ANGULAR MOMENTUM IN QUANTUM MECHANICS —

Consider a quantum system prepared (at a fixed time t,) in the state $|\psi\rangle$ represented by the wave function $\psi(\vec{r}) \equiv \langle \vec{r}|\psi\rangle$

Let $\hat{D}(\vec{n}, \delta\phi)$ an operator describing the PHYSICAL operation of rotating the system by an angle $\delta\phi$ around an axis parallel to the UNIT vector $\vec{n}: |\vec{n}|=1$



$\delta\phi > 0$ if the rotation is counterclockwise.

It is clear that the wave function of the ROTATED system evaluated at the rotated point $\vec{r}'$, must equals the original wave function evaluated at $\vec{r}$, that is:

$$\psi'(\underline{r}') = \psi(\underline{r}) \quad \Leftrightarrow \quad \langle \underline{r}'|\psi'\rangle = \langle \underline{r}|\psi\rangle$$

The RODRIGUES formula tells us how to write $\underline{r}'$ in function of $\underline{r}, \vec{n}$ and $\delta\phi$:

$$\underline{r}' = \underline{r}\cos(\delta\phi) + (\underline{n}\times\underline{r})\sin(\delta\phi) + \underline{n}(\underline{n}\cdot\underline{r})(1-\cos(\delta\phi))$$

If $\delta\phi \ll 1 \Rightarrow \cos(\delta\phi) \simeq 1 - O(\delta\phi^2)$

$\sin(\delta\phi) \simeq \delta\phi$

and

$$\underline{r}' \simeq \underline{r} + (\underline{n}\times\underline{r})\delta\phi + O(\delta\phi^2)$$

we define now $\underline{r}' \equiv \underline{r} + \delta\underline{r}$

where, evidently:

$$\boxed{\delta\underline{r} \equiv (\underline{n}\times\underline{r})\delta\phi}$$

In the equation

$$\psi'(\underline{r}') = \psi(\underline{r})$$

the initial $\underline{r}$ is arbitrary, therefore this equality can be equivalently rewritten either as

$$\psi'(\underline{r}+\delta\underline{r}) = \psi(\underline{r})$$

or

$$\boxed{\psi'(\underline{r}) = \psi(\underline{r}-\delta\underline{r})} \qquad (1.2)$$

Now we use Taylor expansion around $\delta\underline{r} = \underline{0}$ to write

$$\boxed{\psi(\underline{r}-\delta\underline{r}) \simeq \psi(\underline{r}) - \delta\underline{r}\cdot\vec{\nabla}\psi + O(\delta\phi^2)}$$

Since $\delta\underline{r} = (\underline{n}\times\underline{r})\delta\phi$ we can rewrite

$$\delta\underline{r}\cdot\vec{\nabla}\psi = \delta\phi\left\{(\underline{n}\times\underline{r})\cdot\vec{\nabla}\psi\right\}$$

Using the vector identity (Scalar Triple product)

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = \underline{b}\cdot(\underline{c}\times\underline{a}) = \underline{c}\cdot(\underline{a}\times\underline{b})$$

with $\underline{a} = \vec{\nabla}\psi$, $\underline{b} = \underline{n}$ and $\underline{c} = \underline{r}$

we can rewrite:

$$(\underset{\underline{b}}{\underline{n}}\times\underset{\underline{c}}{\underline{r}})\cdot\underset{\underline{a}}{\vec{\nabla}\psi} = \underset{\underline{b}}{\underline{n}}\cdot(\underset{\underline{c}}{\underline{r}}\times\underset{\underline{a}}{\vec{\nabla}\psi})$$

and

$$\delta\underline{r}\cdot\vec{\nabla}\psi = \delta\phi\left[\underline{n}\cdot(\vec{r}\times\vec{\nabla}\psi)\right]$$

Therefore

$$\boxed{\begin{aligned}\psi(\underline{r}-\delta\underline{r}) &\simeq \psi(\underline{r}) - \delta\phi\,\underline{n}\cdot(\vec{r}\times\vec{\nabla}\psi) + \dots\\ &= \left[1-\delta\phi\,\underline{n}\cdot(\vec{r}\times\vec{\nabla})\right]\psi(\underline{r}) + \dots\end{aligned}} \quad (1.3)$$

Now, by hypothesis:

$$|\psi'\rangle = \hat{D}(\vec{n},\delta\phi)|\psi\rangle$$

and

$$\psi'(\underline{r}) = \langle\underline{r}|\psi'\rangle = \langle\underline{r}|\hat{D}(\vec{n},\delta\phi)|\psi\rangle$$

(from (1.2)) $\qquad = \langle\underline{r}-\delta\underline{r}|\psi\rangle$

If $\delta\phi = 0 \Rightarrow \hat{D}(\vec{n},0) = \hat{I}$, therefore, for $\delta\phi \ll 1$

we can write:

$$\hat{D}(\vec{n},\delta\phi) \simeq \hat{I} - i\,\hat{G}_{\underline{n}}\,\delta\phi \quad + O(\delta\phi^2)$$

where $\hat{G}_{\underline{n}}$ is an $\underline{n}$-dependent operator to be determinated, and the imaginary unit "i" is conventional.

From the equality

$$\langle \underline{r}|\,\hat{D}(\vec{n},\delta\phi)\,|\psi\rangle = \langle \underline{r} - \delta\underline{r}|\psi\rangle$$

evaluated at $\delta\phi \ll 1$, it follows that

$$\langle \underline{r}|\,\hat{I} - i\,\hat{G}_{\underline{n}}\,\delta\phi\,|\psi\rangle \simeq \psi(\underline{r} - \delta\underline{r}) \qquad (\text{use } (1.3))$$

$$\Leftrightarrow \langle \underline{r}|\psi\rangle - i\,\delta\phi\,\langle \underline{r}|\,\hat{G}_{\underline{n}}\,|\psi\rangle \simeq \psi(\underline{r}) - \delta\phi\,\underline{n}\cdot(\vec{r}\times\vec{\nabla})\psi(\underline{r})$$

This relation implies:

$$i\,\langle \underline{r}|\,\hat{G}_{\underline{n}}\,|\psi\rangle = \underline{n}\cdot(\vec{r}\times\vec{\nabla})\psi(\underline{r})$$

$$= \left[\vec{e}_x\, n_x\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \vec{e}_y\, n_y\left(-x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right) + \right.$$

$$\left. + \vec{e}_z\, n_z\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right]\langle \underline{r}|\psi\rangle$$

This expression can be rewritten in a more compact form using Einstein's summation convention:

$$\underline{a} \times \underline{b} = \vec{e}_i \underbrace{\varepsilon_{ijk}}_{\text{Totally antisymmetric symbol}} a_j b_k$$

and remembering that:

$$\boxed{\langle \underline{r} | \hat{\underline{P}} | \psi \rangle = -i\hbar \vec{\nabla} \left(\langle \underline{r} | \psi \rangle \right)}$$

Then:

$$\underline{n} \cdot (\vec{r} \times \vec{\nabla} \psi(\underline{r})) = \underline{n} \cdot \left[\vec{r} \times \vec{\nabla} (\langle \underline{r} | \psi \rangle) \right]$$

$$= \frac{1}{-i\hbar} \underline{n} \cdot \left[\underline{r} \times \langle \underline{r} | \hat{\underline{P}} | \psi \rangle \right]$$

However, since $\hat{\underline{r}}$ is Hermitian, then

$$\hat{\underline{r}} | \underline{r} \rangle = \underline{r} | \underline{r} \rangle \Rightarrow \langle \underline{r} | \hat{\underline{r}} = \underline{r} \langle \underline{r} | ,$$

then

$$\underline{r} \times \langle \underline{r} | \hat{\underline{P}} | \psi \rangle = \langle \underline{r} | \hat{\underline{r}} \times \hat{\underline{P}} | \psi \rangle$$

and.

$$\underline{n} \cdot (\vec{r} \times \vec{\nabla}) \psi(\underline{r}) = \frac{i}{\hbar} \underline{n} \cdot \langle \underline{r} | \hat{\underline{r}} \times \hat{\underline{P}} | \psi \rangle$$

Therefore, we have found that:

$$\cancel{i}\langle \underline{r}|\hat{G}_{\underline{n}}|\psi\rangle = \frac{\cancel{i}}{\hbar}\,\underline{n}\cdot\langle \underline{r}|\hat{\underline{r}}\times\hat{\underline{p}}|\psi\rangle$$

Since this must be true for ANY state vector $|\psi\rangle$, we have:

$$\boxed{\hbar\,\hat{G}_{\underline{n}} = \underline{n}\cdot(\hat{\underline{r}}\times\hat{\underline{p}})}$$

So,

$$\boxed{\hat{D}(\vec{n},\delta\phi) \simeq \hat{I} - \frac{i}{\hbar}\delta\phi(\hat{\underline{r}}\times\hat{\underline{p}})\cdot\underline{n}}$$

From <u>classical mechanics</u> we know that the <u>vector</u> $\underline{L} = \underline{r}\times\underline{p}$ is called the ANGULAR MOMENTUM of the system. We borrow this name to call the set of three operators $\hat{\underline{L}} \equiv (\hat{L}_x, \hat{L}_y, \hat{L}_z)$ as:

$$\hat{\underline{L}} = \hat{\underline{r}}\times\hat{\underline{p}}$$

For FINITE rotations by an angle ϕ around a FIXED axis $\vec{n}$, we can imagine $\hat{D}(\vec{n},\phi)$ as a series of N infinitesimal rotations by a small angle $\frac{\phi}{N}$ around the <u>same</u> axis $\vec{n}$ (so, these rotations commute)

$$|\psi'\rangle = \hat{D}(\vec{n},\phi)|\psi\rangle$$

$$= \underset{1}{\hat{D}(\vec{n},\tfrac{\phi}{N})}\,\underset{2}{\hat{D}(\vec{n},\tfrac{\phi}{N})}\cdots\underset{N\text{ times}}{\hat{D}(\vec{n},\tfrac{\phi}{N})}|\psi\rangle$$

$$\simeq \left(\hat{I} - \frac{i}{\hbar}\vec{L}\cdot\vec{n}\,\frac{\phi}{N}\right)^N \xrightarrow[N\to\infty]{} \exp\left(-\frac{i\vec{L}\cdot\vec{n}\,\phi}{\hbar}\right)$$

because rotations around the same angle do commute.

- The operator $\hat{\underline{L}}$ is Hermitian, that is

$$\hat{L}_i^+ = \hat{L}_i \qquad (i = x,y,z)$$

because, for example:

$$\hat{L}_x^+ = (\hat{y}\hat{P}_z - \hat{z}\hat{P}_y)^+$$

$$= (\hat{y}\hat{P}_z)^+ - (\hat{z}\hat{P}_y)^+$$

$$= \hat{P}_z^+\hat{y}^+ - \hat{P}_y^+\hat{z}^+$$

but $\hat{\underline{r}} = \hat{\underline{r}}^+$ and $\hat{\underline{P}} = \hat{\underline{P}}^+$ and $[\hat{x}_k, \hat{P}_\ell] = i\hbar\delta_{k\ell}$ $\Big\} \Rightarrow$

$$\Rightarrow \quad = \hat{y}\hat{P}_z - \hat{z}\hat{P}_y = \hat{L}_x$$ c.v.d

The commutation relations are:

$$[\hat{x}_j, \hat{L}_k] = [\hat{x}_j, \varepsilon_{k\ell m}\hat{x}_\ell\hat{P}_m] = \varepsilon_{k\ell m}\hat{x}_\ell\underbrace{[\hat{x}_j,\hat{P}_m]}_{=i\hbar\delta_{jm}}$$

$$= i\hbar\,\varepsilon_{k\ell j}\hat{x}_\ell$$

$$= i\hbar\,\varepsilon_{jk\ell}\hat{x}_\ell$$

$$[\hat{P}_j, \hat{L}_k] = [\hat{P}_j, \varepsilon_{kem} \hat{X}_e \hat{P}_m]$$

$$= \varepsilon_{kem} \underbrace{[\overset{A}{\hat{P}_j}, \overset{B}{\hat{X}_e} \overset{C}{\hat{P}_m}]}$$ use $[A, BC] = [A,B]C + B[A,C]$

$$= \underbrace{[\hat{P}_j, \hat{X}_e]}_{= -i\hbar \delta_{je}} \hat{P}_m + \hat{X}_e \underbrace{[\hat{P}_j, \hat{P}_m]}_{=0}$$

$$= -i\hbar \varepsilon_{kem} \underbrace{\delta_{je}} \hat{P}_m$$

$$= -i\hbar \varepsilon_{kjm} \hat{P}_m$$

$$= i\hbar \varepsilon_{jkm} \hat{P}_m \Leftrightarrow [\hat{P}_j, \hat{L}_k] = i\hbar \varepsilon_{jkm} \hat{P}_m$$

- Finally, we calculate

$$[\hat{L}_j, \hat{L}_k] = [\varepsilon_{jem} \hat{X}_e \hat{P}_m, \varepsilon_{kst} \hat{X}_s \hat{P}_t]$$

$$= \varepsilon_{jem} \varepsilon_{kst} [\hat{X}_e \hat{P}_m, \hat{X}_s \hat{P}_t]$$

We use:

$$[AB, CD] = A[B,C]D + AC[B,D] + [A,C]DB + C[A,D]B$$

with $A = \hat{X}_e$, $B = \hat{P}_m$, $C = \hat{X}_s$, $D = \hat{P}_t$ $\Rightarrow$

$$\Rightarrow [\hat{X}_e \hat{P}_m, \hat{X}_s \hat{P}_t] = \hat{X}_e \overbrace{[\hat{P}_m, \hat{X}_s]}^{= -i\hbar \delta_{ms}} \hat{P}_t + \hat{X}_e \hat{X}_s \underbrace{[\hat{P}_m, \hat{P}_t]}_{=0} + \underbrace{[\hat{X}_e, \hat{X}_s]}_{=0} \hat{P}_t \hat{P}_m +$$

$$+ \hat{X}_s \underbrace{[\hat{X}_e, \hat{P}_t]}_{= i\hbar \delta_{et}} \hat{P}_m$$

Therefore:

$$[\hat{L}_j, \hat{L}_k] = \varepsilon_{jlm}\,\varepsilon_{kst}\left(-i\hbar\,\delta_{ms}\,\hat{x}_l \hat{P}_t + i\hbar\,\delta_{lt}\,\hat{x}_s \hat{P}_m\right)$$

$$= i\hbar\Big\{-\hat{x}_l \hat{P}_t \underbrace{(\varepsilon_{jlm}\,\varepsilon_{kmt})}_{\substack{= -\varepsilon_{mjl}\,\varepsilon_{mkt} \\ = -(\delta_{jk}\delta_{lt} - \delta_{jt}\delta_{lk})}} + \hat{x}_s \hat{P}_m \underbrace{(\varepsilon_{jlm}\,\varepsilon_{ksl})}_{\substack{= -\varepsilon_{ljm}\,\varepsilon_{lks} \\ = -(\delta_{jk}\delta_{ms} - \delta_{js}\delta_{mk})}}\Big\}$$

$$= -i\hbar\Big\{-\hat{x}_l \hat{P}_t\,\delta_{jk}\delta_{lt} + \hat{x}_l \hat{P}_t\,\delta_{jt}\delta_{lk} + \hat{x}_s \hat{P}_m\,\delta_{jk}\delta_{ms} - \hat{x}_s \hat{P}_m\,\delta_{js}\delta_{mk}\Big\}$$

$$= -i\hbar\Big\{\delta_{jk}\underbrace{\left(-\hat{x}_l \hat{P}_l + \hat{x}_s \hat{P}_s\right)}_{=0} + \hat{x}_k \hat{P}_j - \hat{x}_j \hat{P}_k\Big\}$$

$$= i\hbar\left(\hat{x}_j \hat{P}_k - \hat{x}_k \hat{P}_j\right)$$

This means:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\left(\hat{x}\hat{P}_y - \hat{y}\hat{P}_x\right) = i\hbar\,\hat{L}_z$$

$$[\hat{L}_x, \hat{L}_z] = i\hbar\left(\hat{x}\hat{P}_z - \hat{z}\hat{P}_x\right) = -i\hbar\,\hat{L}_y$$

$$[\hat{L}_y, \hat{L}_z] = i\hbar\left(\hat{y}\hat{P}_z - \hat{z}\hat{P}_y\right) = i\hbar\,\hat{L}_x$$

These relations are equivalent to:

$$\boxed{[\hat{L}_j, \hat{L}_k] = i\hbar\,\varepsilon_{jkl}\,\hat{L}_l}$$

Note: differently from $[\hat{x}_j, \hat{P}_k] = i\hbar\,\delta_{jk}$ here we have an operator

From $\hat{\underline{L}}$ we can build the operator

$$\hat{L}^2 \equiv \hat{\underline{L}} \cdot \hat{\underline{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

This operator commutes with each component of $\hat{\underline{L}}$:

$$[\hat{L}^2, \hat{L}_j] = 0$$

For example:

$$\begin{aligned}
[\hat{L}^2, \hat{L}_z] &= [\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2, \hat{L}_z] \\
&= [\hat{L}_x^2, \hat{L}_z] + [\hat{L}_y^2, \hat{L}_z] \\
&= \hat{L}_x[\hat{L}_x, \hat{L}_z] + [\hat{L}_x, \hat{L}_z]\hat{L}_x + \hat{L}_y[\hat{L}_y, \hat{L}_z] + [\hat{L}_y, \hat{L}_z]\hat{L}_y \\
&= i\hbar\{\hat{L}_x(-i\hbar)\hat{L}_y + (-i\hbar\hat{L}_y)\hat{L}_x + \hat{L}_y(i\hbar\hat{L}_x) + (i\hbar\hat{L}_x)\hat{L}_y\} \\
&= -\hbar^2(\cancel{-\hat{L}_x\hat{L}_y} \cancel{-\hat{L}_y\hat{L}_x} + \cancel{\hat{L}_y\hat{L}_x} + \cancel{\hat{L}_x\hat{L}_y}) \\
&= 0
\end{aligned}$$

and similarly for the others.

Other two useful operators are:

$$\hat{L}_+ \equiv \hat{L}_x + i\hat{L}_y$$

$$\hat{L}_- \equiv \hat{L}_x - i\hat{L}_y$$

and
$$(\hat{L}_+)^\dagger = \hat{L}_x - i\hat{L}_y = \hat{L}_-$$

$$(\hat{L}_-)^\dagger = \hat{L}_x + i\hat{L}_y = \hat{L}_+$$

It is easy to see that:

$$[\hat{L}^2, \hat{L}_\pm] = 0$$

and $[\hat{L}_+, \hat{L}_-] = [L_x + iL_y, L_x - iL_y]$ (From now on I omit the hat)

$$= -i[L_x, L_y] + i[L_y, L_x]$$

$$= -2i[L_x, L_y] = 2\hbar L_z$$

and $[\hat{L}_z, \hat{L}_\pm] = [L_z, L_x \pm iL_y]$

$$= [L_z, L_x] \pm i[L_z, L_y]$$

$$= +i\hbar L_y \pm i(-i\hbar L_x) \qquad (\pm i)(-i) = \begin{cases} +1 \\ -1 \end{cases}$$

$$= \hbar(\pm L_x + iL_y)$$

$$= \hbar \begin{cases} L_+ \\ -L_- \end{cases} = \pm\hbar L_\pm$$

Moreover, it should be noticed that:

add and subtract L_z^2

$$L_+L_- = (L_x + iL_y)(L_x - iL_y)$$

$$= L_x^2 - iL_xL_y + iL_yL_x + L_y^2 + \underbrace{(L_z^2 - L_z^2)}_{=0}$$

$$= L^2 - L_z^2 - i[L_x, L_y]$$

$$= L^2 - L_z^2 - i(i\hbar L_z) = L^2 - L_z^2 + \hbar L_z$$

$$= L^2 - L_z(L_z - I\hbar)$$

– Eigenvalues and eigenvectors –

Since $[L^2, L_z] = 0$ there must be common eigenstates. Let $|\Lambda, \mu\rangle$ denotes such eigenstates, with

$$L^2 |\Lambda,\mu\rangle = \Lambda |\Lambda,\mu\rangle$$

$$L_z |\Lambda,\mu\rangle = \mu |\Lambda,\mu\rangle$$

We choose $|\Lambda,\mu\rangle$ such that

$$\langle \Lambda,\mu | \Lambda',\mu'\rangle = \delta_{\Lambda\Lambda'}\delta_{\mu\mu'}$$

Of course, L_z is NOT special. We could have look for the common eigenstates of, e.g., L^2 and L_x.

It is easy to see that:

$$\Lambda \geq \mu^2$$

because, by definition:

$$\begin{aligned}\Lambda &= \langle \Lambda,\mu | L^2 |\Lambda,\mu\rangle \\ &= \langle \Lambda,\mu | L_x^2 |\Lambda,\mu\rangle + \langle \Lambda,\mu | L_y^2 |\Lambda,\mu\rangle + \mu^2 \\ &= \| L_x |\Lambda,\mu\rangle \|^2 + \| L_y |\Lambda,\mu\rangle\|^2 + \mu^2 \geq \mu^2\end{aligned}$$

because of $L_x = L_x^\dagger$ and $L_y = L_y^\dagger$.

Since $\mu \in \mathbb{R}$ (because $L_z = L_z^\dagger$) then

$$\mu^2 \geqslant 0 \Rightarrow \Lambda \geqslant 0$$

From $\quad \Lambda \geqslant \mu^2 \quad$ it follows that:

if $\mu \geq 0$, then $\mu \leq \Lambda^{1/2}$; if $\mu < 0$, then $\mu = -|\mu|$

and $|\mu| \leq \Lambda^{1/2} \Rightarrow -|\mu| \geqslant -\Lambda^{1/2} \Leftrightarrow \mu \geqslant -\Lambda^{1/2}$

Therefore, we have found that:

$$\boxed{-\Lambda^{1/2} \leq \mu \leq \Lambda^{1/2}}$$

From $[L^2, L_\pm] = 0$ and $[L_z, L_\pm] = \pm\hbar L_\pm$, it follows that

for a given $|\Lambda, \mu\rangle$, we have:

$$L^2 (L_\pm |\Lambda, \mu\rangle) = L_\pm (L^2 |\Lambda, \mu\rangle)$$

$$= \Lambda (L_\pm |\Lambda, \mu\rangle)$$

and

$$L_z (L_\pm |\Lambda, \mu\rangle) = (L_z L_\pm \overbrace{- L_\pm L_z + L_\pm L_z}^{=0}) |\Lambda, \mu\rangle$$

$$= \pm L_\pm |\Lambda, \mu\rangle + \mu (L_\pm |\Lambda, \mu\rangle$$

$$= (\mu \pm 1)(L_\pm |\Lambda, \mu\rangle)$$

Since
$$-\Lambda^{1/2} \leq \mu \leq \Lambda^{1/2}$$
there must exist a minimum and a maximum values μ_{min} and μ_{max}, respectively, of μ for a given Λ, that is:
$$\boxed{-\Lambda^{1/2} \leq \mu_{min} \leq \mu_{max} \leq \Lambda^{1/2}}$$
Then $L_z|\Lambda, \mu_{min}\rangle = \mu_{min}|\Lambda, \mu_{min}\rangle$

and $L_z(L_-|\Lambda,\mu_{min}\rangle) = (\mu_{min}-1)(L_-|\Lambda,\mu_{min}\rangle)$

but it cannot exist an eigenstate $|\Lambda, \mu'\rangle \equiv L_-|\Lambda, \mu_{min}\rangle$ such that $\mu' < \mu_{min}$. Therefore we must have
$$\boxed{L_-|\Lambda, \mu_{min}\rangle = 0}$$
Similarly we see that:
$$\boxed{L_+|\Lambda, \mu_{max}\rangle = 0}$$
We have already found that
$$L_+L_- = L^2 - L_z(L_z - I\hbar) \Rightarrow L^2 = -L_z(-L_z + \hbar I) + L_+L_-$$
Similarly we find that
$$L_-L_+ = (L_x - iL_y)(L_x + iL_y) = L_x^2 + i\underbrace{(L_xL_y - L_yL_x)}_{=i\hbar L_z} + L_y^2 + L_z^2 - L_z^2$$
$$= L^2 - L_z^2 - \hbar L_z \Rightarrow L^2 = L_z(L_z + I\hbar) + L_-L_+$$

We use these two formulas to calculate

$$L^2|\Lambda,\mu_{max}\rangle = L_z(L_z+I\hbar)|\Lambda,\mu_{max}\rangle + \underbrace{L_-L_+|\Lambda,\mu_{max}\rangle}_{=0}$$

$$= \mu_{max}(\mu_{max}+\hbar)|\Lambda,\mu_{max}\rangle$$

However $L^2|\Lambda,\mu_{max}\rangle = \Lambda|\Lambda,\mu_{max}\rangle$ by definition, so it must be:

$$\boxed{\Lambda = \mu_{max}(\mu_{max}+\hbar)} \quad (1.15)$$

Similarly

$$\Lambda|\Lambda,\mu_{min}\rangle = L^2|\Lambda,\mu_{min}\rangle = -L_z(-L_z+\hbar I)|\Lambda,\mu_{min}\rangle + \underbrace{L_+L_-|\Lambda,\mu_{min}\rangle}_{=0}$$

$$= -\mu_{min}(-\mu_{min}+\hbar)|\Lambda,\mu_{min}\rangle$$

So we must have:

$$\begin{cases} \Lambda = \mu_M(\mu_M+\hbar) \\ \Lambda = -\mu_m(-\mu_m+\hbar) \end{cases} \qquad \begin{aligned} \mu_M &\equiv \mu_{max} \\ \mu_m &\equiv \mu_{min} \end{aligned}$$

These two expressions are compatible only if:

$$\mu_M = \hbar l \qquad \text{with } l \geq 0$$
$$\mu_m = -\hbar l$$

and, from (1.15), $\boxed{\Lambda = \hbar^2 l(l+1)}$

All the states between $|\Lambda, \mu_m = -\hbar\ell\rangle$ and $|\Lambda, \mu_M = \ell\rangle$ can be reached applying $L_\pm$:

Suppose to apply L_+ ntimes on $|\Lambda, \mu_m\rangle$

Then $L_z (\underset{1}{L_+} \underset{2}{L_+} \cdots \underset{n}{L_+} |\Lambda, \mu_m\rangle)$

but $[L_z, L_+] = \hbar L_+ \Leftrightarrow$
$$\begin{aligned} L_z L_+ &= L_+ L_z + \hbar L_+ \\ &= L_+ (L_z + \hbar I) \\ &= \hbar L_+ + L_+ L_z \end{aligned}$$

Therefore
$$\begin{aligned} L_z (L_+)^n &= (L_z L_+)(L_+)^{n-1} \\ &= \hbar (L_+)^n + L_+ [L_z (L_+)^{n-1}] \\ &= \hbar (L_+)^n + L_+ \{\hbar (L_+)^{n-1} + L_+ [L_z (L_+)^{n-2}]\} \\ &= 2\hbar (L_+)^n + (L_+)^2 [L_z (L_+)^{n-2}]\} \\ &\vdots \\ &= n\hbar (L_+)^n + (L_+)^n L_z \\ &= (L_+)^n (L_z + n\hbar) \Rightarrow \end{aligned}$$

$$\begin{aligned} \Rightarrow L_z (L_+)^n |\Lambda, \mu_m\rangle &= (L_+)^n (L_z + n\hbar) |\Lambda, \mu_m\rangle \\ &= \hbar (n\hbar + \mu_m)(L_+)^n |\Lambda, \mu_m\rangle \end{aligned}$$

Since we cannot exceed μ_M, it must exist a value n such that:

$$\mu_M = \mu_m + n\hbar$$

but we know that $\mu_m = -\mu_M \Rightarrow \mu_M = -\mu_M + n\hbar \Leftrightarrow$

$$\Leftrightarrow \quad \mu_M = \frac{n}{2}\hbar$$

but $\mu_M = \hbar \ell \geq 0 \Rightarrow \boxed{\ell = \frac{n}{2}} \quad n = 1, 2, 3, \dots$

So, we have found that $\boxed{\ell = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots}$ is either integer or semi-integer!

So our eigenstates are:

$$|\Lambda, \mu\rangle$$

with $\Lambda = \ell(\ell+1)\hbar^2 \qquad \ell \geq 0$

and $\mu = \hbar\underbrace{\{-\ell, -\ell+1, \dots, \ell-1, \ell\}}_{2\ell+1 \text{ values}} \equiv \hbar m \qquad m = -\ell, -\ell+1, \dots, \ell$

– caveat – In many books the ang. mom. is indicated with J. So there are $\hat{J}, \hat{J}_x, \hat{J}_y, \hat{J}_z, \hat{J}_\pm$. The symbol $\hat{L}$ used here is instead reserved to the ORBITAL ang. mom. only that we will study soon.

We have seen that

$$L_\pm |l,m\rangle = C_\pm(l,m)\,|l,m\pm 1\rangle$$

we calculate now the $C_\pm(l,m)$:

$$\| L_\pm |l,m\rangle \|^2 = \langle l,m| L_\mp L_\pm |l,m\rangle = |C_\pm(l,m)|^2 \underbrace{\langle l,m\pm 1|l,m\pm 1\rangle}_{=1 \text{ by hyp}}$$

however

$$\left.\begin{aligned} L_- L_+ &= L^2 - L_z(L_z + I\hbar) \\ L_+ L_- &= L^2 - L_z(L_z - I\hbar) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \langle l,m| L_\mp L_\pm |l,m\rangle = \langle l,m| L^2 - L_z(L_z \pm I\hbar)|l,m\rangle$$

$$= \hbar^2 \left[l(l+1) - m(m\pm 1) \right]$$

We arbitrarily choose the phases such that:

$$\boxed{C_\pm(l,m) = \hbar\sqrt{l(l+1) - m(m\pm 1)}}$$

This relation implies that

$$\langle l',m'| L_\pm |l,m\rangle = C_\pm(l,m) \underbrace{\langle l',m'|l,m\pm 1\rangle}_{=\delta_{l'l}\,\delta_{m',m\pm 1}}$$

$$= \delta_{ll'}\,\delta_{m',m\pm 1}\, C_\pm(l,m)$$

Since
$$L_x = \frac{1}{2}(L_+ + L_-)$$
$$L_y = \frac{1}{2i}(L_+ - L_-)$$

from the formulas for $L_\pm$ we easily obtain

$$\langle l', m' | L_x | l, m\rangle = \frac{\delta_{l'l}}{2}\left[C_+(l,m)\delta_{m',m+1} + C_-(l,m)\delta_{m',m-1}\right]$$

$$\langle l', m' | L_y | l, m\rangle = \frac{\delta_{l'l}}{2i}\left[C_+(l,m)\delta_{m',m+1} - C_-(l,m)\delta_{m',m-1}\right]$$

* Lecture 15 * A. AIELLO-FAU
IK-1 SS2017

– Orbital angular momentum –

Consider again the operator

$$\hat{\underline{L}} = \hat{\underline{r}} \times \hat{\underline{p}}$$

$$= -i\hbar\, \underline{r} \times \vec{\underline{\nabla}} \quad \text{in position representation.}$$

In spherical coordinates (r, θ, ϕ) with:

$$0 \leq r < \infty; \quad 0 \leq \theta \leq \pi; \quad 0 \leq \phi < 2\pi$$

we have:

$$x = r\sin\theta\cos\phi \quad , y = r\sin\theta\sin\phi \ , \ z = r\cos\theta$$

and
$$\vec{\nabla}\psi = \vec{e}_r \frac{\partial\psi}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \vec{e}_\phi \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\underline{r} = r\vec{e}_r$$

Then
$$\underline{r} \times \vec{\nabla}\psi = r(\vec{e}_r \times \vec{\nabla}\psi)$$

$$= \cancel{r}\frac{1}{\cancel{r}}\frac{\partial\psi}{\partial\theta}(\vec{e}_r \times \vec{e}_\theta) + \cancel{r}\frac{1}{\cancel{r}\sin\theta}\frac{\partial\psi}{\partial\phi}(\vec{e}_r \times \vec{e}_\phi)$$

where
$$\begin{cases} \vec{e}_r = \dfrac{\partial\vec{r}}{\partial r} = \vec{e}_1 \sin\theta\cos\phi + \vec{e}_2 \sin\theta\sin\phi + \vec{e}_3\cos\theta \\ \vec{e}_\theta = \dfrac{1}{r}\dfrac{\partial\vec{r}}{\partial\theta} = \vec{e}_1\cos\theta\cos\phi + \vec{e}_2\cos\theta\sin\phi - \vec{e}_3\sin\theta \\ \vec{e}_\phi = \dfrac{1}{r\sin\theta}\dfrac{\partial\vec{r}}{\partial\phi} = -\vec{e}_1\sin\phi + \vec{e}_2\cos\phi \end{cases}$$

Therefore

$$\vec{e}_r \times \vec{e}_\theta = (\vec{e}_1 \sin\theta\cos\phi + \vec{e}_2 \sin\theta\sin\phi + \vec{e}_3 \cos\theta) \times$$

$$\times (\vec{e}_1 \cos\theta\cos\phi + \vec{e}_2 \cos\theta\sin\phi - \vec{e}_3 \sin\theta)$$

$$= \cancel{\vec{e}_3 \sin\theta\cos\theta\sin\phi\cos\phi} + \vec{e}_2 \sin^2\theta\cos\phi +$$

$$- \cancel{\vec{e}_3 \sin\theta\cos\theta\sin\phi\cos\phi} - \vec{e}_1 \sin^2\theta\sin\phi +$$

$$+ \vec{e}_2 \cos^2\theta\cos\phi - \vec{e}_1 \cos^2\theta\sin\phi$$

$$= \vec{e}_2 \cos\phi - \vec{e}_1 \sin\phi = \vec{e}_\phi$$

Similarly:

$$\vec{e}_r \times \vec{e}_\phi = -\vec{e}_\theta$$

← cyclic property

and

$$\vec{r} \times \vec{\nabla}\psi = \vec{e}_\phi \frac{\partial\psi}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}\vec{e}_\theta$$

So, there is no radial component as expected

Replacing back $\vec{e}_\phi$ and $\vec{e}_\theta$ we get:

$$\vec{r} \times \vec{\nabla}\psi = (-\vec{e}_1 \sin\phi + \vec{e}_2 \cos\phi)\frac{\partial\psi}{\partial\theta} - (\vec{e}_1 \cos\theta\cos\phi + \vec{e}_2 \cos\theta\sin\phi +$$

$$- \vec{e}_3 \sin\theta)\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$= \vec{e}_1\left(-\sin\phi\frac{\partial\psi}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial\psi}{\partial\phi}\right) + \vec{e}_2\left(\cos\phi\frac{\partial\psi}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial\psi}{\partial\phi}\right)$$

$$+ \vec{e}_3 \frac{\partial\psi}{\partial\phi}$$

Therefore:

$$\begin{cases} \hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \\ \hat{L}_y = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \\ \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \end{cases}$$

and

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$$

$$= i\hbar\left\{\frac{\partial}{\partial\theta}\underbrace{(\sin\phi \mp i\cos\phi)}_{= \mp i(\cos\phi \pm i\sin\phi)} + \cot\theta\frac{\partial}{\partial\phi}(\cos\phi \pm i\sin\phi)\right\}$$

$$= i\hbar e^{\pm i\phi}\left(\mp i\frac{\partial}{\partial\theta} + \cot\theta\frac{\partial}{\partial\phi}\right)$$

$$= \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

It should be noticed that

$$\hat{\underline{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$= -\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]$$

$$= \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - r^2\nabla_{\underline{r}}^2$$

expressed in spherical coordinates

- eigenfunctions of $\hat{L}_z$

$$\hat{L}_z \psi = \lambda \psi \Leftrightarrow -i\hbar \frac{\partial \psi}{\partial \phi} = \lambda \psi \Rightarrow$$

$$\Rightarrow \psi(r,\theta,\phi) = f(r,\theta) \exp\left(i \frac{\lambda}{\hbar} \phi\right)$$

with $f(r,\theta)$ arbitrary.
The function $\psi(r,\theta,\phi)$ is single-valued only if it is periodic in ϕ. This implies that:

$$\frac{\lambda}{\hbar} = \text{integer}$$

We call $m = 0, \pm 1, \pm 2, \ldots$ such integer, so $\lambda = m\hbar$

and
$$\boxed{\psi(r,\theta,\phi) = f(r,\theta) \exp(im\phi)}$$

If we define
$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} \exp(im\phi)$$

Then
$$\int_0^{2\pi} \Phi_m^*(\phi)\, \Phi_{m'}(\phi)\, d\phi = \delta_{mm'}$$