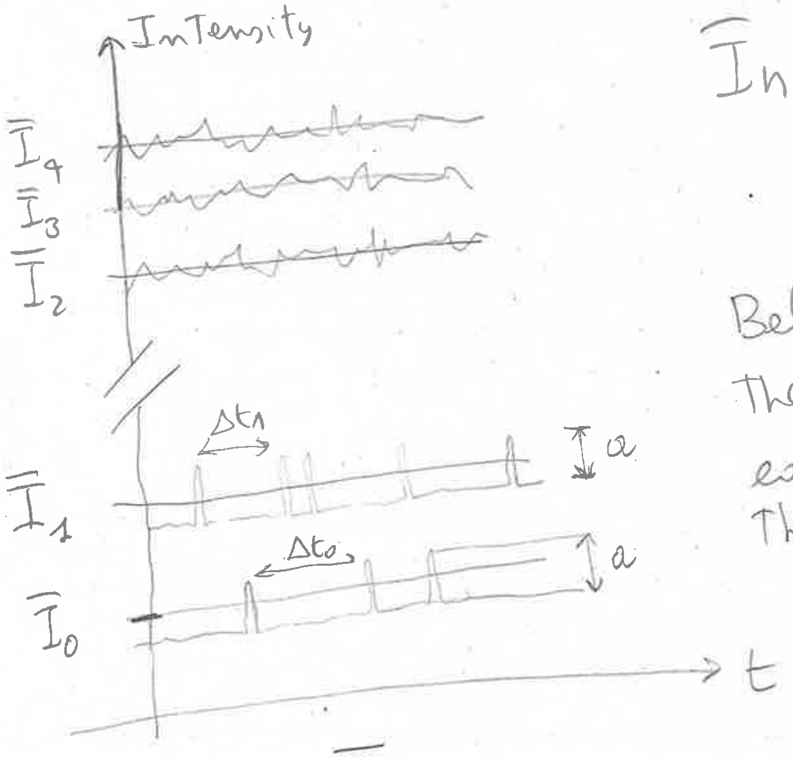
* Lecture 16 * A. AIELLO

FAU IK-1 SS2017

It is an EXPERIMENTAL evidence that light possesses energy that, at sufficiently low intensity, is observed in DISCRETE amounts, which cannot be further split. (Intensity = Energy per unit of time)

A typical record of the intensity of a BRIGHT beam of light, looks like this:



$\bar{I}_n$ = average measured intensity

Below some threshold, the amplitude a of each "spike" remain the same although $\bar{I}_1 > \bar{I}_2$

$\overline{\Delta t_1} < \overline{\Delta t_0}$

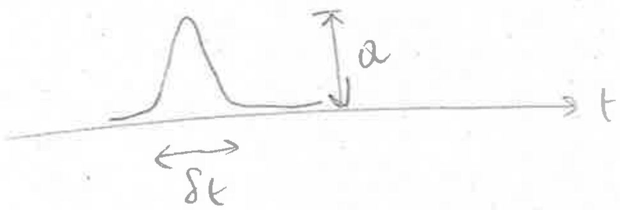
In a time interval T, there are about $\frac{T}{\Delta t_1} > \frac{T}{\Delta t_0}$ spikes of amplitude a. Then

$$\bar{I}_1 \propto \frac{T}{\Delta t_1} a > \bar{I}_0 = \frac{T}{\Delta t_0} \cdot a$$

It is an experimental fact that if the light has an angular frequency ω, then each spike carries an energy $\hbar\omega$.

This energy cannot be split. In fact, it has NEVER been observed an amount of energy $\hbar\omega/2$ in a beam of light of frequency ω

If the duration of each spike is about δt



and there are, on average,

$$N = \frac{T}{\Delta t}$$

spikes on a time interval T, the total energy (average) recorded during T is:

$$\bar{\mathcal{E}}_T = N\hbar\omega = \frac{T}{\Delta t}\hbar\omega$$

and the intensity (average) is

$$\bar{I}_T = \frac{\bar{\mathcal{E}}_T}{T} = \frac{\hbar\omega}{\Delta t}$$

The carrier of each spike of energy, is called a "PHOTON" or, QUANTUM OF LIGHT.

- Consider an electromagnetic plane wave traveling in the z-direction. This is a solution of Maxwell's equations in vacuum,

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad 1.2\,a)$$

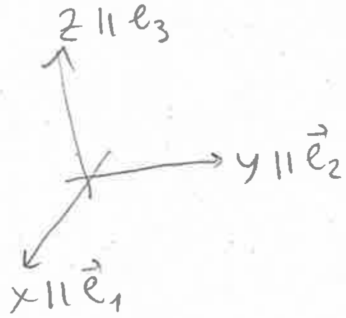
$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad 1.2\,b)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad 1.2\,c)$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = 0 \qquad 1.2\,d)$$

which depends on z-coordinate only

$$\vec{E} = \vec{E}(z,t) \; ; \quad \vec{B} = \vec{B}(z,t)$$



such that:

$$\begin{cases} \vec{E} = \vec{e}_1 E_x(z,t) + \vec{e}_2 E_y(z,t) \\ \vec{B} = \vec{e}_1 B_x(z,t) + \vec{e}_2 B_y(z,t) \end{cases}$$

with

$$\begin{cases} E_x(z,t) = E_{0x}\cos(kz - \omega t + \varphi_x) \\ E_y(z,t) = E_{0y}\cos(kz - \omega t + \varphi_y) \end{cases}$$

with

$$kc = \omega \qquad k = \frac{2\pi}{\lambda}, \; \lambda = \text{wavelength of light}$$

and

$$\vec{B} = \frac{1}{c}\vec{e}_3 \times \vec{E}$$

$$= +\frac{1}{c}\left[-\vec{e}_1 E_{0y}\cos(kz - \omega t + \varphi_y) + \vec{e}_2 E_{0x}\cos(kz - \omega t + \varphi_x)\right]$$

It is convenient to introduce a complex-number notation:

$$E_x(z,t) = \mathrm{Re}\left[A_{0x} e^{i(kz-\omega t)}\right]$$

$$E_y(z,t) = \mathrm{Re}\left[A_{0y} e^{i(kz-\omega t)}\right]$$

where
$$\begin{cases} A_{0x} = E_{0x}\, e^{i\varphi_x} \\ A_{0y} = E_{0y}\, e^{i\varphi_y} \end{cases}$$

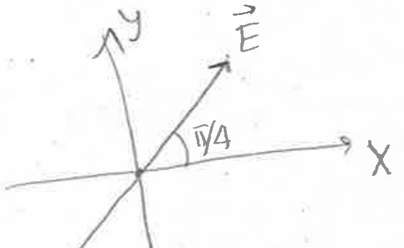
From now on we will work only with complex amplitudes, remembering to take the real part at the end of calculations.

The POLARIZATION state of light is determined by the vector $\vec{A} = \vec{e}_1 A_{0x} + \vec{e}_2 A_{0y}$. For example:

1) if $A_{0y} = 0$, the wave is LINEARLY polarized in the x-direction, that is the tip of the electric-field vector oscillate with angular frequency ω along the x-axis.

2) if $A_{0x} = 0$, the wave is linearly polarized in the y-direction

3) if $A_{0x} = A_{0y}$ the wave is linearly polarized at 45°.



4) if $A_{0y} = e^{i\pi/2} A_{0x} = i A_{0x}$, then the y-component of $\vec{E}$ has a 90° phase lag with respect to the x-component. The wave is said to be RIGHT CIRCULARLY POLARIZED:

$E_x(z,t) = E_{0x}\cos(kz-\omega t+\varphi)$; $E_y(z,t) = E_{0y}\cos\left(kz-\omega t+\varphi+\frac{\pi}{2}\right)$

5) Similarly, if $A_{0y} = -i A_{0x}$, the wave is LEFT circularly polarized

— The ENERGY DENSITY of an electromagnetic field is:

$$u(z,t) = \frac{\epsilon_0}{2}\left(\vec{E}\cdot\vec{E} + c^2\vec{B}\cdot\vec{B}\right) = \frac{1}{2}\left(\epsilon_0\,\vec{E}\cdot\vec{E} + \frac{1}{\mu_0}\vec{B}\cdot\vec{B}\right)$$

$$\begin{cases} \epsilon_0 = \text{permittivity of vacuum} \simeq 8.854\ldots \times 10^{-12}\ \frac{F}{m} \\ \mu_0 = \text{vacuum permeability} \simeq 4\pi\cdot 10^{-7}\ N/A^2 \\ \frac{1}{\epsilon_0\mu_0} = c^2 \end{cases}$$

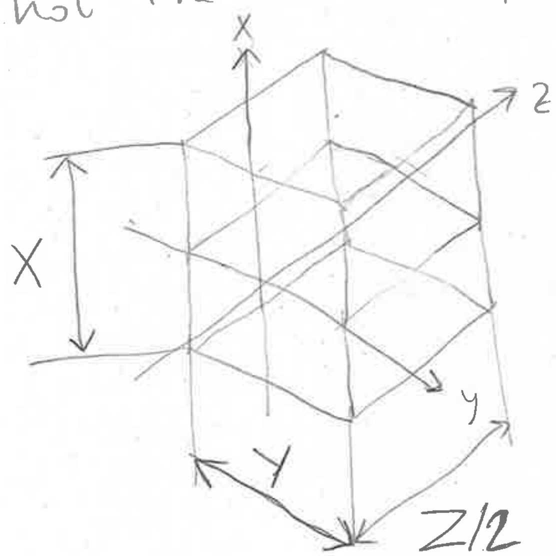
For our plane wave, since $\vec{B} = \frac{1}{c}\vec{e}_3 \times \vec{E} \Rightarrow$

$$\Rightarrow c^2\,\vec{B}\times\vec{B} = (\vec{e}_3\times\vec{E})\cdot(\vec{e}_3\times\vec{E})$$
$$= \vec{E}\cdot\vec{E}$$

Therefore:

$$u(z,t) = \epsilon_0\,\vec{E}\cdot\vec{E}$$
$$= \epsilon_0\left[E_{0x}^2\cos^2(kz-\omega t+\varphi_x) + E_{0y}^2\cos^2(kz-\omega t+\varphi_y)\right]$$
$$= \left(|A_{0x}|^2 + |A_{0y}|^2\right)\epsilon_0 = \epsilon_0|\vec{A}|^2$$

Suppose that the wave occupy a volume V:



$V = XYZ$

with $X, Y, Z \gg \lambda$

The total energy of the wave is then:

$$\mathcal{E} = \int_V u(z,t)\,dx\,dy\,dz = XY\int_Z u(z,t)\,dz$$

$$= XYZ\left\{ E_{0x}^2 \frac{1}{Z}\int_{-Z/2}^{Z/2} \cos^2(kz-\omega t+\varphi_x)\,dz + \right.$$

$$\left. + E_{0y}^2 \frac{1}{Z}\int_{-Z/2}^{Z/2} \cos^2(kz-\omega t+\varphi_y)\,dz \right\}\epsilon_0$$

$$= V\epsilon_0\left\{ E_{0x}^2\left[\frac{1}{2} + \underbrace{\frac{\sin(kZ)\cos(2\varphi_x-2\omega t)}{2kZ}}_{\simeq 0 \text{ for } kZ\gg 1}\right] + E_{0y}^2\left[\frac{1}{2} + \underbrace{\frac{\sin(kZ)\cos(2\varphi_y-2\omega t)}{2kZ}}_{\simeq 0 \text{ for } kZ\gg 1}\right]\right\}$$

$$\cong \frac{V}{2}\left(E_{0x}^2 + E_{0y}^2\right)\epsilon_0 = \frac{V|\vec{A}|^2}{2}\epsilon_0 = \frac{\epsilon_0}{2} V \vec{A}^*\cdot\vec{A} = \frac{\epsilon_0}{2} V \langle A|A\rangle$$

↑ equivalent notation

- Now, consider a POLAROID FILTER, that is a plate that transmits light polarized along some direction $\vec{n} = n_x \vec{e}_1 + n_y \vec{e}_2$, with $|\vec{n}| = 1$

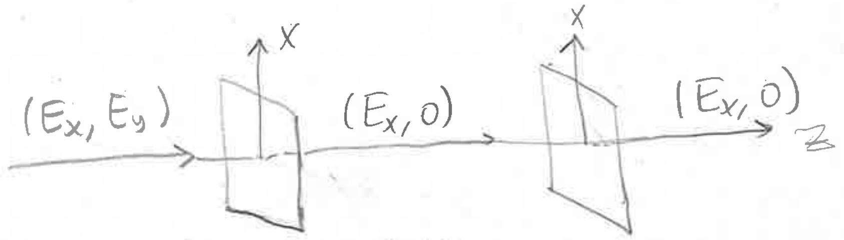
Suppose to have a 45° polarized wave: $A_{0x} = A_{0y} \equiv A$.

$$\vec{E}^{inc} = A e^{i(kz-\omega t)}(\vec{e}_1 + \vec{e}_2) \qquad inc = \text{incident field}$$

and that the polaroid filter is oriented such that

$$\vec{n} = \vec{e}_1$$

Then, the Transmitted field will be polarized along x because if I put a second polarizer parallel to x, the light will be transmitted:



polaroid filter ∥ x

The Transmitted field is:

$$\vec{E}^{trans} = \vec{e}_1 (\vec{e}_1 \cdot \vec{E}) = \vec{e}_1 A e^{i(kz-\omega t)}$$

The energy of the incident field is

$$\mathcal{E}^{inc} = V \cdot |A|^2$$

The energy of the Transmitted field is

$$\mathcal{E}^{trans} = V \frac{|A|^2}{2}$$

So, the energy is divided by two. How can we describe this phenomenon if the light is made of discrete and INDIVISIBLE quanta? How the energy can be split?

We know from <u>experiments</u> that the energy of the incident wave is carried by <u>photons</u> each of energy $\hbar\omega$. Therefore

$$\mathcal{E}^{inc} = N^I \hbar\omega = V|A|^2 \qquad N = 1, 2, 3, \ldots$$

This means that in the incident wave there are, on overall,

$$N^I = \frac{V|A|^2}{\hbar\omega} \quad \text{photons}$$

Then, since
$$\frac{\mathcal{E}^{Trans}}{\mathcal{E}^{inc}} = \frac{1}{2}$$

we will have an average number N^T of Transmitted photons:
$$N^T = \frac{N^I}{2}$$

So, only <u>half</u> of the photon will go through the polaroid filter (also called: polarizer). Therefore, if $N^I = 1$, the $N^T = 1/2$ means that sometime the incident photon will be Transmitted an sometime it will not. More precisely, if we repeat the experiment many times, the photon will be Transmitted about 50% of the times.

In NON-RELATIVISTIC QUANTUM MECHANICS, the PROBABILITY CURRENT $\vec{J}$ (or probability FLUX) of a single particle of mass m prepared in the state $\Psi(\underline{r},t)$, is
$$\vec{J}(\underline{r},t) = \frac{\hbar}{m} \operatorname{Im}(\Psi^* \vec{\nabla} \Psi)$$

It is possible to demonstrate (see any Textbook on classical electrodynamics), that for an electromagnetic wave of energy density
$$u(\underline{r},t) = \frac{1}{2}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2\right)$$

the following CONTINUITY EQUATION holds

$$\boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0}$$

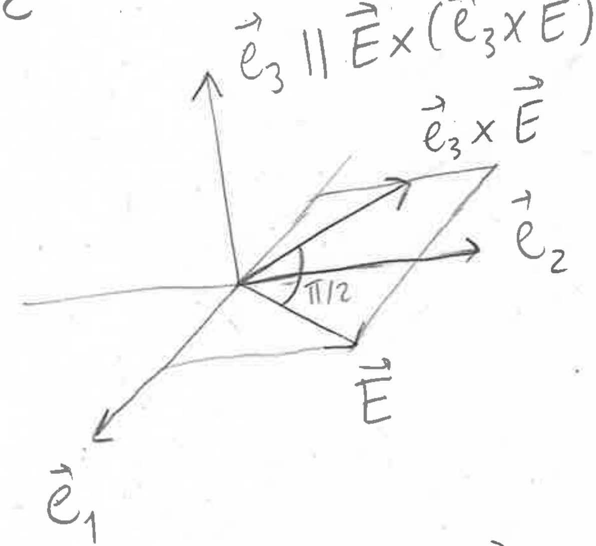
where the "current"

$$\boxed{\vec{S}(\underline{r},t) \equiv \frac{1}{\mu_0}\,\vec{E}(\underline{r},t) \times \vec{B}(\underline{r},t)}$$

is called the <u>POYNTING VECTOR</u> of the wave.

For our plane wave:

$$\vec{E}\times\vec{B} = \frac{1}{c}\,\vec{E}\times(\vec{e}_3\times\vec{E})$$



$$\vec{e}_3\times\vec{E} = \vec{e}_3\times(\vec{e}_1 E_x + \vec{e}_2 E_y) = \vec{e}_2 E_x - \vec{e}_1 E_y$$

$$\vec{E}\times(\vec{e}_3\times\vec{E}) = (\vec{e}_1 E_x + \vec{e}_2 E_y)\times(-\vec{e}_1 E_y + \vec{e}_2 E_x)$$

$$= \vec{e}_3 E_x^2 + \vec{e}_3 E_y^2$$

Therefore

$$\vec{S} = \frac{\epsilon_0}{\mu_0 \epsilon_0} \frac{1}{c} \vec{e}_3 \left(E_x^2 + E_y^2\right)$$

$$= \vec{e}_3 \, c \underbrace{\left(\epsilon_0 \vec{E} \cdot \vec{E}\right)}_{= \text{energy density } u(\underline{r},t)} = \vec{e}_3 \, c \, u(\underline{r},t)$$

So

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = \epsilon_0 \frac{\partial}{\partial t}(\vec{E} \cdot \vec{E}) + c\epsilon_0 \frac{\partial}{\partial z}(\vec{E} \cdot \vec{E})$$

$$= 0$$

because

$$\frac{\partial}{\partial t} \cos^2(kz - \omega t + \varphi) = 2\omega \sin(kz - \omega t + \varphi) \cos(kz - \omega t + \varphi)$$

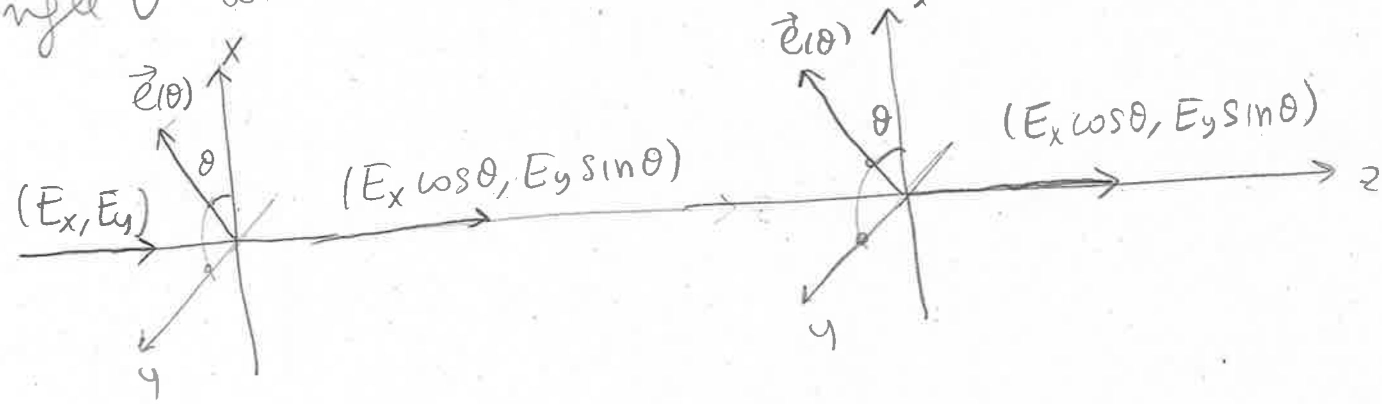
$$\frac{\partial}{\partial z} \cos^2(kz - \omega t + \varphi) = -2k \sin(kz - \omega t + \varphi) \cos(kz - \omega t + \varphi)$$

and $kc = \omega$ (dispersion relation of vacuum)

In summary

$$\boxed{\vec{S} = \vec{e}_3 \, c \, u(\underline{r},t)}$$

So, if I have a poloroid filter at an angle θ to the x-axis



$$\vec{e}(\theta) = \vec{e}_1 \cos\theta + \vec{e}_2 \sin\theta$$

the Transmitted field will be:

$$\vec{E}^{\text{Trans}} = \vec{e}(\theta)\left[\vec{e}(\theta)\cdot\vec{E}^{inc}\right]$$

$$= \vec{e}(\theta)\left(E_x \cos\theta + E_y \sin\theta\right)$$

Then, the TRANSMISSION COEFFICIENT of the poloroid can be calculated as follows. During the time interval $0 \le t \le \tau$, an energy

$$\mathcal{E}^{inc} = (XYZ)\,\frac{1}{Z}\int_0^{Z/c} u(\underline{r},t)\,dt$$

$$\simeq \frac{V}{c}\,\frac{\epsilon_0}{2}\left(E_{0x}^2 + E_{0y}^2\right)$$

will impinge onto the poloroid.

Similarly, the Transmitted energy during τ:

$$\mathcal{E}^{trans} \simeq \frac{V}{c}\frac{\epsilon_0}{2}\left(E_{0x}^2\cos^2\theta + E_{0y}^2\sin^2\theta\right)$$

therefore:

$$T = \frac{\text{Transmitted flux}}{\text{Incident flux}}$$

$$= \frac{c\,\mathcal{E}^{trans}}{c\,\mathcal{E}^{inc}} = \frac{|A_x|^2}{|A_x|^2+|A_y|^2}\cos^2\theta + \frac{|A_y|^2}{|A_x|^2+|A_y|^2}\sin^2\theta$$

— Single photon formalism —

A single photon of frequency ω contained in a volume V, has energy (from page 6)

$$\mathcal{E} = \frac{\epsilon_0}{2} V \vec{A}^* \cdot \vec{A} = \underset{\substack{\uparrow \\ \text{one photon}}}{1} \cdot \hbar\omega \Rightarrow |A_x|^2 + |A_y|^2 = \frac{2\hbar\omega}{\epsilon_0 V}$$

Then, we DEFINE the <u>state vector</u> $|\psi\rangle$ of the photon polarization as:

$$|\psi\rangle = (\psi_x, \psi_y) \doteq \underset{\substack{\uparrow \\ \text{matrix representation}}}{\begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix}} \qquad (12.1)$$

where

$$\begin{cases} \psi_x \equiv \sqrt{\dfrac{\epsilon_0 V}{2\hbar\omega}}\, A_x \\ \psi_y \equiv \sqrt{\dfrac{\epsilon_0 V}{2\hbar\omega}}\, A_y \end{cases} \qquad (12.2)$$

In this way:

$$\|\psi\|^2 = \langle\psi|\psi\rangle = |\psi_x|^2 + |\psi_y|^2$$

$$= \frac{\epsilon_0 V}{2\hbar\omega}\left(|A_x|^2 + |A_y|^2\right)$$

$$= 1$$

The $|\psi\rangle$ vectors are vectors in $\mathbb{C}^2$.

Examples:

$$|\psi\rangle = |x\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} : \quad \text{x-polarization}$$

$$|\psi\rangle = |y\rangle \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix} : \quad \text{y-polarization}$$

$$|\psi\rangle = |R\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}; \quad \text{Right circular polarization.}$$

$$|\psi\rangle = |L\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}; \quad \text{Left circular polarization}$$

As usual:

$$\langle\psi| = (|\psi\rangle)^* \doteq [\psi_x^* \;\; \psi_y^*]$$

and

$$\langle\phi|\psi\rangle = [\phi_x^* \;\; \phi_y^*]\begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix} = \phi_x^*\psi_x + \phi_y^*\psi_y$$

The basis vectors $|x\rangle$ and $|y\rangle$ are orthogonal.

$$\langle x|y\rangle = 0$$

and complete:

$$|x\rangle\langle x| + |y\rangle\langle y| \doteq \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2 \quad \text{(2x2 identity matrix)}$$

It should be noticed that:

$$|R\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\i & -i\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \Leftrightarrow |R\rangle = U|x\rangle$$

$$|L\rangle \doteq \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\i & -i\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \Leftrightarrow |L\rangle = U|y\rangle$$

The matrix $\quad U = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\i & -i\end{bmatrix}$

is unitary:

$$UU^{\dagger} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\i & -i\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\1 & i\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Therefore:

$$\langle R|R\rangle = \langle x|U^{+}U|x\rangle = \langle x|x\rangle = 1$$

and $\langle R|L\rangle = \langle x|U^{+}U|y\rangle = \langle x|y\rangle = 0$

and $$|R\rangle\langle R| + |L\rangle\langle L| = U\left(|x\rangle\langle x| + |y\rangle\langle y|\right)U^{+}$$
$$= U\,I\,U^{+}$$
$$= I$$

Any state of polarization $|\psi\rangle$ can be written as

$$|\psi\rangle = \psi_x|x\rangle + \psi_y|y\rangle$$
$$= \psi_x \frac{|R\rangle + |L\rangle}{\sqrt{2}} + \psi_y \frac{|R\rangle - |L\rangle}{\sqrt{2}\,i}$$
$$= |R\rangle \frac{\psi_x - i\psi_y}{\sqrt{2}} + |L\rangle \frac{\psi_x + i\psi_y}{\sqrt{2}}$$
$$\equiv |R\rangle \psi_R + |L\rangle \psi_L$$

The quantity

$$\langle x|\psi\rangle = \psi_x \underbrace{\langle x|x\rangle}_{=1} + \psi_y \underbrace{\langle x|y\rangle}_{=0} = \psi_x$$

is called the PROBABILITY AMPLITUDE that a photon prepared in the polarization state $|\psi\rangle$ will be transmitted across a polaroid filter parallel to the x-axis.

Why? Because, classically we know that:

$$\frac{\text{Energy Transmitted by polaroid at } x}{\text{Energy incident}} = \frac{|A_x|^2}{|A_x|^2+|A_y|^2}$$

$$\text{from 12.2} \quad = \frac{|\psi_x|^2}{|\psi_x|^2+|\psi_y|^2}$$

$$= \frac{|\langle x|\psi\rangle|^2}{\langle\psi|\psi\rangle}$$

$$= |\langle x|\psi\rangle|^2$$

$$= \text{Probability}$$

Similarly $\langle R|\psi\rangle$ is the probability amplitude that a photon in the polarization state $|\psi\rangle$ will be Transmitted across a polarizer that (classically) Transmits only circularly polarized light.

$$\langle R|\psi\rangle = \psi_x \langle R|x\rangle + \psi_y \langle R|y\rangle = \frac{1}{\sqrt{2}}(\psi_x + i\psi_y)$$

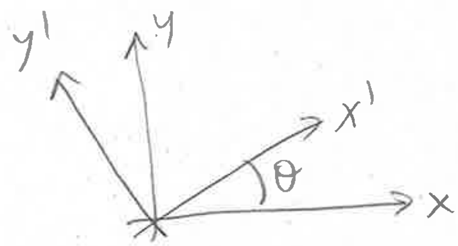
$$\text{Probability} = |\langle R|\psi\rangle|^2$$

$$= \frac{1}{2}(\psi_x^* - i\psi_y^*)(\psi_x + i\psi_y)$$

$$= \frac{1}{2}\left(|\psi_x|^2 + |\psi_y|^2 + i\psi_x^*\psi_y - i\psi_y^*\psi_x\right)$$

$$= \frac{1}{2}\left[1 + 2\,\underbrace{\mathrm{Im}(\psi_x\psi_y^*)}_{\text{"interference" term}}\right]$$

– Transformation of bases –



Consider the two basis in figure. Then

$$|\psi\rangle = \psi_x |x\rangle + \psi_y |y\rangle$$
$$= \psi_{x'} |x'\rangle + \psi_{y'} |y'\rangle \qquad \text{where} \begin{cases} \psi_{x'} = \langle x' | \psi\rangle \\ \psi_{y'} = \langle y' | \psi \rangle \end{cases}$$

that is:

$$\psi_{x'} = \langle x' | \psi \rangle = \psi_x \langle x' | x \rangle + \psi_y \langle y' | y \rangle$$
$$\psi_{y'} = \langle y' | \psi \rangle = \psi_x \langle y' | x \rangle + \psi_y \langle y' | y \rangle$$

$$\Leftrightarrow \begin{bmatrix} \psi_{x'} \\ \psi_{y'} \end{bmatrix} = \underbrace{\begin{bmatrix} \langle x' | x \rangle & \langle y' | y \rangle \\ \langle y' | x \rangle & \langle y' | y \rangle \end{bmatrix}}_{\text{Transformation matrix from the } x,y \text{ basis to the } x',y' \text{ basis.}} \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix}$$

In the case of figure:

$$\begin{cases} |x'\rangle = \cos\theta |x\rangle + \sin\theta |y\rangle \\ |y'\rangle = -\sin\theta |x\rangle + \cos\theta |y\rangle \end{cases}$$

and

Transf. matrix $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Now, instead, consider the NEW state $|\psi'\rangle$ obtained rotating the polarization by an angle θ:

$$|\psi\rangle \rightarrow |\psi'\rangle = \hat{D}(\theta)|\psi\rangle$$

where, by definition

$$\hat{D}(\theta)|x\rangle = |x'\rangle$$

$$\hat{D}(\theta)|y\rangle = |y'\rangle$$

Then

$$\psi'_x \equiv \langle x|\psi'\rangle = \langle x|\hat{D}|\psi\rangle = \langle x|\hat{D}\,\hat{I}\,|\psi\rangle$$

$$= \underbrace{\langle x|\hat{D}|x\rangle}_{=\langle x|x'\rangle}\underbrace{\langle x|\psi\rangle}_{=\psi_x} + \underbrace{\langle x|\hat{D}|y\rangle}_{=\langle x|y'\rangle}\underbrace{\langle y|\psi\rangle}_{=\psi_y}$$

$$= \psi_x\langle x|x'\rangle + \psi_y\langle x|y'\rangle$$

$$= \psi_x\cos\theta - \psi_y\sin\theta$$

Similarly:

$$\psi'_y = \langle y|\psi'\rangle = \langle y|\hat{D}|\psi\rangle$$

$$= \langle y|x'\rangle\psi_x + \langle y|y'\rangle\psi_y$$

$$= \psi_x\sin\theta + \psi_y\cos\theta$$

In matrix form:

according to p 17

$$\begin{bmatrix}\psi'_x \\ \psi'_y\end{bmatrix} = \begin{bmatrix}\langle x|x'\rangle & \langle x|y'\rangle \\ \langle y|x'\rangle & \langle y|y'\rangle\end{bmatrix}\begin{bmatrix}\psi_x \\ \psi_y\end{bmatrix} = R^{+}(\theta)\begin{bmatrix}\psi_x \\ \psi_y\end{bmatrix}$$

SUMMARIZING:

PASSIVE rotation: $\begin{bmatrix}\psi_x \\ \psi_y\end{bmatrix} \to \begin{bmatrix}\psi_{x'} \\ \psi_{y'}\end{bmatrix} = R(\theta)\begin{bmatrix}\psi_x \\ \psi_y\end{bmatrix}$

ACTIVE rotation: $\begin{bmatrix}\psi_x \\ \psi_y\end{bmatrix} \to \begin{bmatrix}\psi'_x \\ \psi'_y\end{bmatrix} = R^{+}(\theta)\begin{bmatrix}\psi_x \\ \psi_y\end{bmatrix}$

Are there polarization states INVARIANT under rotations?
In other words, find the states $|\phi\rangle$ defined by

$$\hat{D}(\theta)|\phi\rangle = \lambda_\phi |\phi\rangle$$

In the x, y basis this relation becomes:

$$\langle x|\hat{D}|\phi\rangle = \lambda_\phi \langle x|\phi\rangle \Leftrightarrow \overbrace{\hat{D}_{xx}}^{=\langle x|D|x\rangle = \langle x|x'\rangle}\langle x|\phi\rangle + \overbrace{D_{xy}}^{=\langle x|D|y\rangle = \langle x|y'\rangle}\langle y|\phi\rangle = \lambda_\phi \underbrace{\langle x|\phi\rangle}_{\equiv \phi_x}$$

$$\langle y|\hat{D}|\phi\rangle = \lambda_\phi \langle y|\phi\rangle \Leftrightarrow \underbrace{D_{yx}}_{=\langle y|D|x\rangle = \langle y|x'\rangle}\langle x|\phi\rangle + \underbrace{D_{yy}}_{=\langle y|D|y\rangle = \langle y|y'\rangle}\langle y|\phi\rangle = \lambda_\phi \underbrace{\langle y|\phi\rangle}_{\equiv \phi_y}$$

In matrix form:

$$\underbrace{\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}}\begin{bmatrix}\sigma_x \\ \sigma_y\end{bmatrix} = \lambda_\sigma \begin{bmatrix}\sigma_x \\ \sigma_y\end{bmatrix}$$

$$= \cos\theta\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} - \sin\theta\begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}$$

$$= \cos\theta\, I_2 - i\sin\theta\, S \quad \text{where} \quad S \equiv \begin{bmatrix}0 & -i \\ i & 0\end{bmatrix}$$

Since $[S, I_2] = 0$, it is enough to find the eigenstates of S

Notice that

$$S^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

therefore, if

$$S|\sigma\rangle = \lambda_\sigma |\sigma\rangle$$

Then

$$\underbrace{|\sigma\rangle = S^2|\sigma\rangle = \lambda_\sigma S|\sigma\rangle = \lambda_\sigma^2|\sigma\rangle}_{\Rightarrow \lambda_\sigma^2 = 1}$$

Then $\boxed{\lambda_\pm = \pm 1}$

A straightforward calculation shows that..

$$S|R\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} = +|R\rangle$$

$$S|L\rangle = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ i \end{bmatrix} = -|L\rangle$$

So, we can rewrite

$$S|\pm\rangle = \pm|\pm\rangle$$

where $|+\rangle = |R\rangle$ and $|-\rangle = |L\rangle$

Therefore:

$$D|\pm\rangle = (\cos\theta\,\hat{I} - i\sin\theta\, S)|\pm\rangle$$

$$= (\cos\theta \mp i\sin\theta)|\pm\rangle = e^{\mp i\theta}|\pm\rangle$$

So the eigenvalue equation:

$$\hat{D}(\theta)|\phi\rangle = \lambda_\phi|\phi\rangle$$

becomes:

$$\hat{D}(\theta)|\pm\rangle = e^{\mp i\theta}|\pm\rangle$$

this means that the polarization states with <u>circular</u> polarization are invariant (they take only a phase factor) under rotation.

We call the generator $\hat{S}$: $\hat{S} \doteq \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \equiv S$ in the x,y basis

the <u>SPIN OPERATOR FOR THE PHOTON</u>

We will see soon that $\hbar$ times the spin of a photon is the component of the angular momentum in the z-direction of an electromagnetic wave propagating in the z-direction.

- Homeworks -

1) Given the polarization basis $|x\rangle, |y\rangle$ calculate:

a) the state vectors of a photon linearly polarized at $\pm 45°$ with respect to the x-axis. Show that the matrix connecting these states, denoted $|\pm 45\rangle$, with $|x\rangle, |y\rangle$ is unitary.

b) The matrix, in the basis $|x\rangle, |y\rangle$, of a polaroid filter that transmits light y-polarized. Show that this matrix is a projector

c) As point b), but the polarizer transmits light with polarization parallel to the axis x': $\vec{e}_{x'} = \cos\theta\, \vec{e}_x + \sin\theta\, \vec{e}_y$

d) As point b), but the polarizer transmits only R-polarized light

e) A "λ/n plate" is a transparent anysotropic plate that introduce a phase diffence between the x and y components of the electric field, equal to $\Delta\phi = \frac{2\pi}{n}$: $(E_x, E_y) \rightarrow (E_x e^{i\phi_x}, E_y e^{i\phi_y})$; $\Delta\phi = \phi_y - \phi_x$

Write, in the $|x\rangle, |y\rangle$ basis, the matrix for a $\lambda/4$ and a $\lambda/2$ plate