In the last lecture we have considered electromagnetic waves of the form

$$\vec{E}(z,t) = \vec{e}_1 E_x(z,t) + \vec{e}_2 E_y(z,t)$$
$$= \vec{e}_1 E_1 \cos(kz - \omega t + \varphi_1) + \vec{e}_2 E_2 \cos(kz - \omega t + \varphi_2)$$

and

$$\vec{B}(z,t) = \vec{e}_1 B_x(z,t) + \vec{e}_2 B_y(z,t)$$
$$= \vec{e}_1 B_1 \cos(kz - \omega t + \phi_1) + \vec{e}_2 B_2 \cos(kz - \omega t + \phi_2)$$

where
$$\begin{cases} B_1 = -E_2 \\ \phi_1 = \varphi_2 \end{cases} \quad \text{and} \quad \begin{cases} B_2 = E_1 \\ \phi_2 = \varphi_1 \end{cases}$$

Then we have calculated the Energy Density

$$u(z,t) = \frac{\epsilon_0}{2}\left(\vec{E}\cdot\vec{E} + c^2 \vec{B}\cdot\vec{B}\right)$$
$$= \epsilon_0 \left[E_1^2 \cos^2(kz - \omega t + \varphi_1) + E_2^2 \cos^2(kz - \omega t + \varphi_2)\right]$$

and the Poynting vector

$$\vec{S} = c^2 \epsilon_0 \vec{E} \times \vec{B}$$
$$= \vec{e}_3 \, c \, u(z,t)$$

Since, $\dfrac{\partial u}{\partial t} = \epsilon_0 \omega\, v(z,t)$

$(kc=\omega)$ $\quad \dfrac{\partial u}{\partial z} = -\epsilon_0 k\, v(z,t) = -\dfrac{1}{c}\left[\epsilon_0 \omega\, v(z,t)\right],$

where

$$v(z,t) \equiv E_1^2 \sin\left[2(kz-\omega t+\varphi_1)\right] + E_2^2 \sin\left[2(kz-\omega t+\varphi_2)\right],$$

then it is clear that u and $\vec{S}$ fullfill the following continuity equation:

$$\boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = 0}$$

The Poynting vector represents the flow of electromagnetic energy associated with a travelling wave. In the SI system it has units of $\dfrac{\text{Watt}}{m^2}$.

At optical frequencies ($\sim 10^{15}$ Hz), $\vec{S}$ is a very rapidly function of time (twice as rapid as the fields), therefore its instantaneous value would be an impractical quantity to measure directly. This suggests that we employ an averaging procedure over some time interval T that accounts for the

(finite, as opposed to instantaneous) time resolu- (17-3)
tion of the detector.

The time-averaged value of some function $f(t)$ over an interval T is:

$$\langle f(t)\rangle_T = \frac{1}{T}\int_{t-T/2}^{t+T/2} f(t)\,dt$$

It is not difficult to calculate

$$\boxed{\langle \cos^2(\alpha-\omega t)\rangle_T = \frac{1}{2} + \cos[2(\alpha-\omega t)]\,\frac{\sin(\omega T)}{2\omega T}}$$

and

$$\boxed{\langle \cos(\alpha-\omega t)\rangle_T = \cos(\alpha-\omega t)\,\frac{\sin(\omega T/2)}{\omega T/2}}$$

At optical frequencies $\omega \approx 10^{15}$ Hz, if our detector averages over 1 microsecond $= 10^{-6}$ s, we have

$$\omega T \sim 10^{15}\cdot 10^{-6} = 10^{-9}$$

and

$$\left|\frac{\sin(\omega T)}{2\omega T}\right| \simeq 3\times 10^{-10} \qquad \left|\frac{\sin(\omega T/2)}{\omega T/2}\right| \simeq 6\times 10^{-10}$$

Therefore, we can write:

$$\begin{cases} \langle \cos^2(\alpha - \omega t)\rangle_T \simeq \frac{1}{2} \\ \\ \langle \cos(\alpha - \omega t)\rangle_T \simeq 0 \end{cases}$$

It should be noticed that these results can be achieved exactly if we choose T equal to one cycle of oscillation, that is:

$$T = \frac{2\pi}{\omega}$$

In this case, we talk about "cycle-averaged" quantities. A useful theorem is the following:

Let $A, B \in \mathbb{C}$ be complex numbers and $a(t), b(t) \in \mathbb{R}$ two real functions with harmonic time-dependence, defined by:

$$a(t) = \mathrm{Re}[A e^{-i\omega t}], \quad b(t) = \mathrm{Re}[B e^{-i\omega t}], \quad \begin{cases} \omega \in \mathbb{R} \\ A, B \in \mathbb{C} \end{cases}$$

then:

$$\boxed{\langle a(t)\, b(t)\rangle_{T = \frac{2\pi}{\omega}} = \frac{1}{2}\mathrm{Re}[A B^*]}$$

proof.

$$2a(t) = A e^{-i\omega t} + A^* e^{i\omega t}$$

$$2b(t) = B e^{-i\omega t} + B^* e^{i\omega t}$$

$$\langle a(t)\, b(t) \rangle_{T=\frac{2\pi}{\omega}} = \frac{\omega}{2\pi} \int_{t-\pi/\omega}^{t+\pi/\omega} \Big[\frac{1}{2}\left(A e^{-i\omega t'} + A^* e^{i\omega t'}\right) \times \frac{1}{2}\left(B e^{-i\omega t'} + B^* e^{i\omega t'}\right) \Big] dt'$$

$$= \frac{\omega}{8\pi} \int_{t-\pi/\omega}^{t+\pi/\omega} \Big[\left(AB e^{-i2\omega t'} + A^* B^* e^{2i\omega t'}\right) + \left(AB^* + A^* B\right) \Big] dt'$$

$$= \frac{\cancel{\omega}}{48\pi} \left(AB^* + A^* B\right) \frac{2\pi}{\cancel{\omega}}$$

$$= \frac{1}{2} \, \frac{AB^* + A^* B}{2} = \frac{1}{2} \mathrm{Re}\left(AB^*\right)$$

c.v.d.

Suppose now that the wave has a cross-sectional area A on the xy plane, and that it has a longitudinal extension L in the z direction. Therefore, the wave is contained in a volume $V = A \cdot L$

The cycle-averaged (c.a.) energy density is:

$$\langle u(z,t)\rangle_{T=\frac{2\pi}{\omega}} = \frac{\epsilon_0}{2}(E_1^2 + E_2^2)$$

and the total c.a. energy H in the volume V is:

$$H = \frac{\epsilon_0 V}{2}(E_1^2 + E_2^2)$$

If we agree to write:

$$\vec{E}_c(z,t) \equiv e^{i(kz-\omega t)}\left[\vec{e}_1 E_1 e^{i\varphi_1} + \vec{e}_2 E_2 e^{i\varphi_2}\right]$$

Then

$$\vec{E}(z,t) = \mathrm{Re}\left[\vec{E}_c(z,t)\right]$$

If we further define:

$$A_1 \equiv E_1 e^{i\varphi_1} \quad , \quad A_2 \equiv E_2 e^{i\varphi_2}$$

Then

$$\begin{cases} \vec{E}_c(z,t) = e^{i(kz-\omega t)}(\vec{e}_1 A_1 + \vec{e}_2 A_2) \\ \vec{B}_c(z,t) = \frac{1}{c} e^{i(kz-\omega t)}(-\vec{e}_1 A_2 + \vec{e}_2 A_1) \end{cases}$$

Therefore, it is clear that the c.a. energy density (which from now on we denote with $\langle f(t)\rangle_{T=\frac{2\pi}{\omega}} \equiv \overline{f(t)}$) can be written as:

$$\overline{u(z,t)} = \frac{\epsilon_0}{4}(\vec{E}_c^* \cdot \vec{E}_c + c^2 \vec{B}_c^* \cdot \vec{B}_c)$$

$$= \frac{\epsilon_0}{4}\Big\{(\vec{e}_1 A_1^* + \vec{e}_2 A_2^*)\cdot(\vec{e}_1 A_1 + \vec{e}_2 A_2) +$$

$$+ (-\vec{e}_1 A_2^* + \vec{e}_2 A_1)\cdot(-\vec{e}_1 A_2 + \vec{e}_2 A_1)\Big\}$$

$$= \frac{\epsilon_0}{4}\left(|A_1|^2 + |A_2|^2 + |A_2|^2 + |A_1|^2\right) \qquad (\text{remember: } \vec{e}_i \cdot \vec{e}_j = \delta_{ij})$$

$$= \frac{\epsilon_0}{2}\left(|A_1|^2 + |A_2|^2\right) = \frac{\epsilon_0}{2}\left(E_1^2 + E_2^2\right)$$

Similarly, the c.a. Poynting vector is calculated as:

$$\overline{\vec{S}(z,t)} = \frac{\epsilon_0}{2} c^2 \,\mathrm{Re}\,(\vec{E}_c^* \times \vec{B}_c)$$

$$= \vec{e}_3 \, c \, \overline{u(z,t)}$$

Suppose that there is only one (1) photon of energy $\hbar\omega$ in the volume V. Therefore:

$$\hbar\omega = H = \frac{\epsilon_0 V}{2}\left(|A_1|^2 + |A_2|^2\right)$$

Therefore, if we define the complex amplitudes:

$$\Psi_1 = \sqrt{\frac{\epsilon_0 V}{2\hbar\omega}}\, A_1 \quad \text{and} \quad \Psi_2 = \sqrt{\frac{\epsilon_0 V}{2\hbar\omega}}\, A_2$$

Then: $|\Psi_1|^2 + |\Psi_2|^2 = 1$

We define the state vector of a single photon (17-8)
of frequency ω :

$$|\psi\rangle \doteq \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$$\|\psi\|^2 = \langle\psi|\psi\rangle = 1$$

Last time we have found eigenvalues and eigenstates of the rotation operator $\hat{D}(\theta)$
In the $\{|x\rangle, |y\rangle\}$ basis

$$\hat{D}(\theta) \doteq \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \equiv R(\theta)$$

Since $\qquad R(\theta) = \exp\left(-i\theta\sigma_2\right)$

where $\qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2^{nd}$ Pauli Matrix

then we introduce the SPIN OF THE PHOTON operator
$\hat{S}$: $\qquad \hat{S} \doteq \hbar S \quad$ in $\{|x\rangle, |y\rangle\}$ basis

and $\qquad \hat{D}(\theta) = \exp\left(-\frac{i}{\hbar}\theta\hat{S}\right) \leftarrow$ basis-independent expression

Amplitude mechanics

Given $|\psi\rangle$ we can choose a basis, say $\{|R\rangle, |L\rangle\}$, and write

$$|\psi\rangle = |R\rangle\langle R|\psi\rangle + |L\rangle\langle L|\psi\rangle$$

Since
$$\hat{S}|R\rangle = +\hbar|R\rangle$$
$$\hat{S}|L\rangle = -\hbar|L\rangle$$

Then $|\langle R|\psi\rangle|^2$ = Probability that a measurement of the spin of the photon will give $+\hbar$ as a result

Since a measurement of $\hat{S}$ will produce always either $+\hbar$ or $-\hbar$, one is tempted to infer that actually the photon is already (prior to the measurement) either in the state $|R\rangle$ with probability $\alpha = |\langle R|\psi\rangle|^2$ or in the state $|L\rangle$ with probability $1-\alpha = |\langle L|\psi\rangle|^2$

But if this were true, you could not explain why an x-polarized photon in state $|x\rangle$ will never be found behind a polarizer oriented at y, because

$$P(y|x) = |\langle y|x\rangle|^2 = 0$$

According to our (wrong) reasoning, if an x-polarized photon is either in the state $|R\rangle$ or in the state $|L\rangle$, then

$$P(y|x) = \text{probability of } y \text{ given } x$$

would be:

$$P(y|x) = P(y|R)P(R|x) + P(y|L)P(L|x)$$
$$= |\langle y|R\rangle|^2 |\langle R|x\rangle|^2 + |\langle y|L\rangle|^2 |\langle L|x\rangle|^2$$
$$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$$

If we use the jargon:

$$\langle\phi|\psi\rangle \equiv a(\phi|\psi) \in \mathbb{C}$$
$$= \text{amplitude of } \phi \text{ given } \psi$$

then the correct calculation amounts to sum the amplitudes before taking the modulus square.

$$a(y|x) = a(y|R)a(R|x) + a(y|L)a(L|x)$$
$$= \langle y|R\rangle\langle R|x\rangle + \langle y|L\rangle\langle L|x\rangle$$
$$= \langle y|\Big(\underbrace{|R\rangle\langle R| + |L\rangle\langle L|}_{=\mathbb{1}}\Big)|x\rangle$$
$$= \langle y|x\rangle = 0 \quad \leftarrow \text{correct result}$$

Therefore, in general (for example in the $\{|x\rangle, |y\rangle\}$ and $\{|R\rangle, |L\rangle\}$ basis)

$$P(\phi|\psi) = \left| a(\phi|R)a(R|\psi) + a(\phi|L)a(L|\psi)\right|^2$$
$$= \left| a(\phi|x)a(x|\psi) + a(\phi|y)a(y|\psi)\right|^2$$

In the specific case

$$P(y|x) = |a(y|x)|^2$$

$$= |\langle y|R\rangle\langle R|x\rangle + \langle y|L\rangle\langle L|x\rangle|^2$$

$$= (\langle x|R\rangle\langle R|y\rangle + \langle x|L\rangle\langle L|y\rangle)(\langle y|R\rangle\langle R|x\rangle + \langle y|L\rangle\langle L|x\rangle)^2$$

$$= |\langle R|x\rangle|^2 |\langle y|R\rangle|^2 + |\langle L|x\rangle|^2 |\langle y|L\rangle|^2 +$$

$$+ \underbrace{\langle x|R\rangle\langle R|y\rangle\langle y|L\rangle\langle L|x\rangle + \langle x|L\rangle\langle L|y\rangle\langle y|R\rangle\langle R|x\rangle}_{\text{Interference Terms (I.T.)}}$$

$$= \underbrace{P(y|R)\,P(R|x) + P(y|L)\,P(L|y)}_{\text{old wrong result}} + I.T. = 0$$

So, the interference terms are necessary to guarantee the correct result. We have thus learned the basic rules for the composition of <u>probability amplitudes</u>:

1. If "1" and "2" are two consecutive events, the amplitude for the event 2 given 1 is:

$$a(2|1) = a(2)\,a(1)$$

$a(i)$ = probability amplitude that event "i" occurs

For example, if event 1 is that a photon x-polarized is transmitted across a R-polarizer, and event 2 is that this photon (now in the state $|R\rangle$) is transmitted across a y-polarizer, then

$$a(1) = \langle R|x\rangle$$
$$a(2) = \langle y|R\rangle$$

$$a(2|1) = \langle y|R\rangle\langle R|x\rangle$$

2. The amplitude for a process that can occur in several mutually exclusive but indistinguishable ways, is the sum of the amplitudes for each of the individual ways:

$$a(n|1) = \sum_{k=2}^{n-1} a(n|k)a(k|1)$$

For example: $a(y|x) = a(y|R)a(R|x) + a(y|L)a(L|x)$

3. The total probability is the modulus square of the total amplitude:

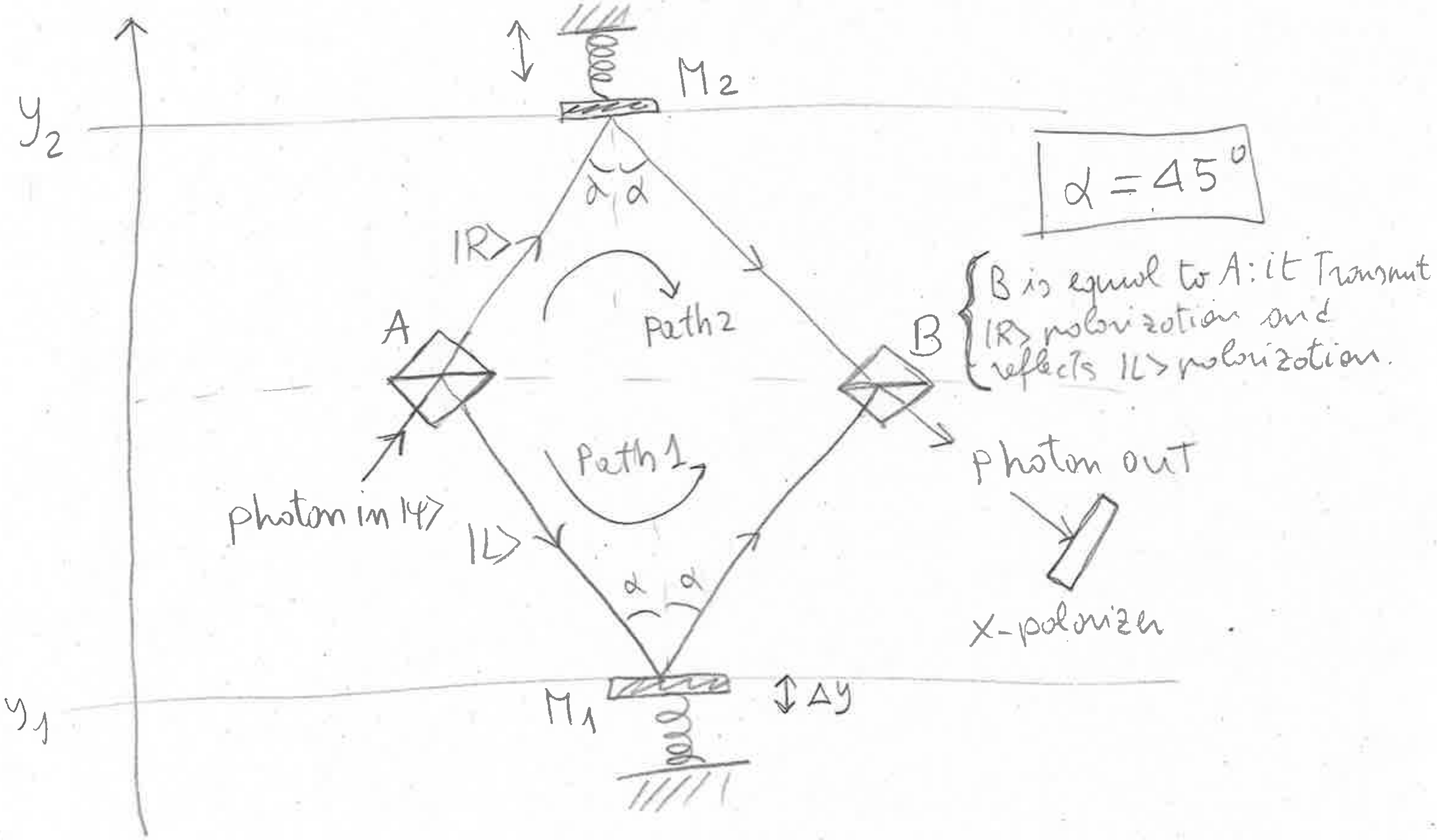
$$P(\psi_{fin}|\psi_{in}) = |a(\psi_{fin}|\psi_{in})|^2$$

$$= \Big|\sum_{k \in \text{all the ways}} a(\psi_{fin}|k)a(k|\psi_{in})\Big|^2$$

Here "indistinguishable" means that if we try to distinguish the alternative ways, we destroy interference.

- Example -

Consider this experimental arrangement:



A photon of frequency ω and linear momentum

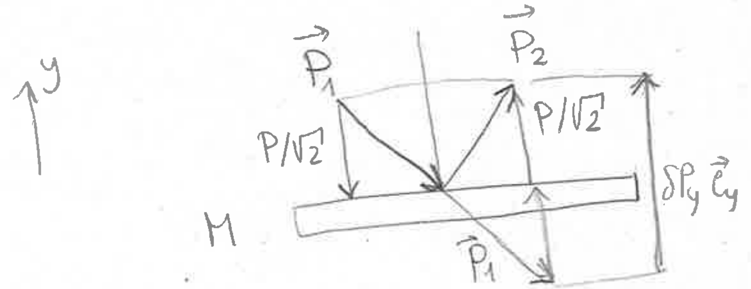
$$P = \frac{\hbar 2\pi}{\lambda} \quad / \quad \lambda = \text{wavelenght}$$

is prepared in the $|\psi\rangle$ state of polarization.
It enters the device A that either transmits it in the state $|R\rangle$ or reflects it in the state $|L\rangle$.
The two mirrors M_1 and M_2 can move along the y-axis

After reflection, the two PATHS are recombined by the device B. The lenght of the two paths is the same:

$$\overline{AM_1} + \overline{M_1B} = \overline{AM_2} + \overline{M_2B} \equiv L$$

We cannot say that the photon Toke path 1 or 2, but if we can measure the recoil of mirrors M_1 and M_2 we can infer the path followed by the photon. When the photon hits the mirror, it Transfers momentum $\sqrt{2}\,p$ to the mirror in the y-direction:



$$\vec{P}_2 = \vec{P}_1 + 2\frac{P}{\sqrt{2}}\vec{e}_y = \vec{P}_1 + \sqrt{2}\,P\,\vec{e}_y$$

To measure this variation $\delta P_y = \sqrt{2}\,P$, we need to know the momentum of the mirror with an uncertainty ΔP_y less than δP_y:

$$\Delta P_y < \delta P_y = \sqrt{2}\,P = \frac{\sqrt{2}\,2\pi\hbar}{\lambda}$$

However, from Heisenberg uncertainty principle

$$\Delta P_y\, \Delta y \geqslant 2\pi\hbar$$

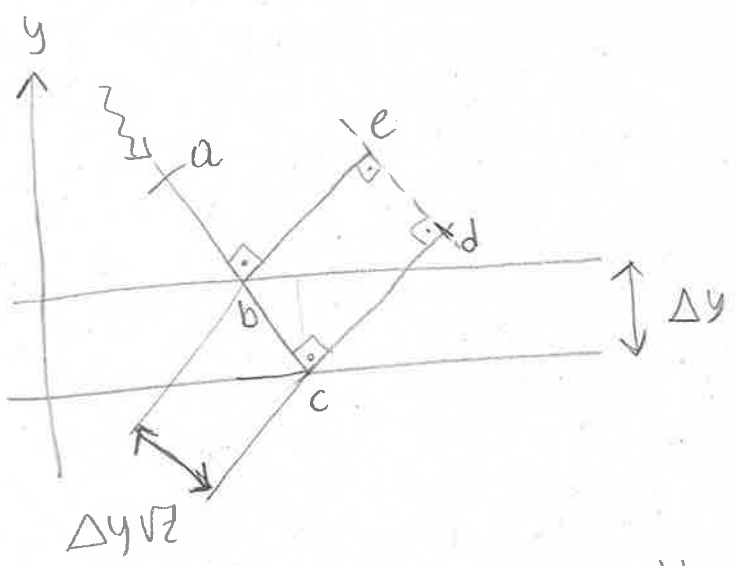
That is, to resolve such a Tiny momentum variation δP_y, the position of the mirror in the y-direction must be uncertain by an amount:

$$\Delta P_y < \delta P_y \iff \delta P_y \, \Delta y > \Delta P_y \, \Delta y \geq 2\pi\hbar$$

$$\Rightarrow \qquad \Delta y > \frac{2\pi\hbar}{\delta P_y} = \frac{\cancel{2\pi\hbar}}{\cancel{2\pi\hbar}} \frac{1}{\frac{\sqrt{2}}{\lambda}} = \frac{\lambda}{\sqrt{2}}$$

But if the uncertainty Δy of the mirror is $\lambda/\sqrt{2}$, the uncertainty ΔL on the length of the path is:

$$\Delta L = \lambda$$



$\overline{be} = \overline{cd}$ by construction

$$\Delta L = (\overline{ac} + \overline{cd}) - (\overline{ab} + \overline{be}) = \overline{bc}$$

$$\overline{bc} = \Delta y \sqrt{2} = \frac{\lambda}{\cancel{\sqrt{2}}} \cancel{\sqrt{2}} = \lambda$$

An uncertainty ΔL on the path, means

$$e^{ikz} \to e^{ik(z+\Delta L)} = e^{ikz + i\delta\phi}$$

an uncertainty $\delta\phi = k\Delta L = \frac{2\pi}{\cancel{\lambda}} \cancel{\lambda} = 2\pi$ on the phase

but $\delta\phi = 2\pi$ means <u>Total random phase!</u>

So if recoil of M_1 gives ϕ_L and recoil of mirror M_2 gives ϕ_R, then the <u>recombined</u> beam at B will be: $|\psi'\rangle = |R\rangle\langle R|\psi\rangle e^{i\phi_R} + |L\rangle\langle L|\psi\rangle e^{i\phi_L}$

In summary:

- I enter with a photon in state $|\psi\rangle = |\psi_{in}\rangle$
- It splits in two paths:

$$|\psi\rangle = \underbrace{|R\rangle\langle R|\psi\rangle}_{\text{path 2}} + \underbrace{|L\rangle\langle L|\psi\rangle}_{\text{path 1}}$$

- It is recombined in $|\psi_{out}\rangle$

$$|\psi'\rangle = |R\rangle\langle R|\psi\rangle e^{i\phi_R} + |L\rangle\langle L|\psi\rangle e^{i\phi_L} = |\psi_{out}\rangle$$

If no measurement-induced recoil $\phi_R = 0 = \phi_L$ and

$$|\psi'\rangle = |\psi\rangle$$

Therefore, if I put an x-polarizer I have:

$$P(x|\psi) = |\langle x|\psi_{out}\rangle|^2$$

$$= \begin{cases} |\langle x|\psi\rangle|^2 & \text{no-measurement} \\ |\langle x|\psi'\rangle|^2 & \text{with-measurement} \end{cases}$$

where

$$|\langle x|\psi'\rangle|^2 = |\underbrace{\langle x|R\rangle}_{=1/\sqrt{2}}\underbrace{\langle R|\psi\rangle}_{\equiv a(R)} e^{i\phi_R} + \underbrace{\langle x|L\rangle}_{=1/\sqrt{2}}\underbrace{\langle L|\psi\rangle}_{\equiv a(L)} e^{i\phi_L}|^2$$

$$= \frac{1}{2}\left[a^*(R)e^{-i\phi_R} + a^*(L)e^{-i\phi_L}\right]\left[a(R)e^{i\phi_R} + a(L)e^{i\phi_L}\right]$$

$$= \frac{1}{2}\left[|a(R)|^2 + |a(L)|^2 + a^*(R)a(L)e^{-i(\phi_R-\phi_L)} + a^*(L)a(R)e^{i(\phi_R-\phi_L)}\right]$$

If we repeat many times the experiment, the uncertainties on the phases ϕ_R and ϕ_L is 2π. The ensemble average will be therefore 0:

$$\overline{e^{\pm i(\phi_R-\phi_L)}} = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{\pm i\alpha} = 0$$

This cancels the interference terms! Therefore the OBSERVED probability as a result of many measurements, is:

$$\overline{|\langle x|\psi'\rangle|^2} \Rightarrow \frac{1}{2}\left(|\langle R|\psi\rangle|^2 + |\langle L|\psi\rangle|^2\right)$$

ensemble average

which is the same result as if the photon was <u>either</u> in the state $|R\rangle$ <u>or</u> in the state $|L\rangle$

- So, if the two paths becomes <u>distinguishable</u> because of the measurement, I do not have to sum the amplitudes any more, but the probabilities, as in the classical case.
- This loss of interference does not depend on the fact of <u>watching or not watching</u> the mirrors, but from the way we have prepared them, irrespective from the actual observation.

– Unpolarized light and density matrix –

Consider two different polarization states for a photon, say $|\psi_1\rangle$ and $|\psi_2\rangle$, with $\langle\psi_1|\psi_2\rangle = 0$
We can build the coherent superposition state

$$\langle\psi_1|\psi_2\rangle = 0$$

$(\alpha, \beta \in \mathbb{C})$
$$|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$$

Such a kind of state is called a PURE state.
Each photon prepared in $|\psi\rangle$ has a probability $|\alpha|^2$ and $|\beta|^2$ to behave, under measurement, as $|\psi_1\rangle$

or $|\psi_2\rangle$.

Now, suppose to have TWO INDEPENDENT single-photon sources; S_1 and S_2. S_1 emits photons in $|\psi_1\rangle$ and S_2 emits photons in $|\psi_2\rangle$. The rate of emission of S_1 and S_2 is such that in the unit of time there is a probability P_1 to get a photon from S_1 and a probability P_2 to get a photon from S_2.
By hypothesis, in the unit of time I always get a photon, therefore

$$P_1 + P_2 = 1$$

So, we may ask: given a photon (from either S_1 or S_2), what is the probability that will be transmitted by an x-polarizer?

If we had only one source S emitting photons in $|\psi\rangle$, the answer would be:

$$|\langle x|\psi\rangle|^2 = |\alpha\langle x|\psi_1\rangle + \beta\langle x|\psi_2\rangle|^2$$

but now the prob. is $|\langle x|\psi_1\rangle|^2$ if emitted by S_1 and $|\langle x|\psi_2\rangle|^2$ if emitted by S_2. Therefore:

$$\left.\begin{array}{l}\text{Probability that a}\\ \text{photon emitted either}\\ \text{from } S_1 \text{ or } S_2 \text{ will pass}\\ \text{across an x-polarizer}\end{array}\right\} = P_1|\langle x|\psi_1\rangle|^2 + P_2|\langle x|\psi_2\rangle|^2$$

$$= P_1\langle x|\psi_1\rangle\langle\psi_1|x\rangle + P_2\langle x|\psi_2\rangle\langle\psi_2|x\rangle$$

$$= \langle x|\Big(P_1|\psi_1\rangle\langle\psi_1| + P_2|\psi_2\rangle\langle\psi_2|\Big)|x\rangle$$

$$\equiv \langle x|\hat{\rho}|x\rangle$$

where $$\hat{\rho} \equiv P_1|\psi_1\rangle\langle\psi_1| + P_2|\psi_2\rangle\langle\psi_2|$$

is called: <u>density operator</u> for <u>MIXED</u> states.

It should be noticed that also $|\langle x|\psi\rangle|^2$ can be written as:

$$|\langle x|\psi\rangle|^2 = \langle x|\underbrace{(|\psi\rangle\langle\psi|)}_{\equiv\hat{\rho}_{|\psi\rangle} = \text{density matrix for a pure state.}}|x\rangle$$

In a pure state, being $\hat{\rho}_{|\psi\rangle}$ a projector, we have:

$$\left(\hat{\rho}_{|\psi\rangle}\right)^2 = |\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{=1}\langle\psi|$$

$$= |\psi\rangle\langle\psi| = \hat{\rho}_{|\psi\rangle}$$

Instead for a mixed state

$$\hat{\rho}^2 = \left(P_1|\psi_1\rangle\langle\psi_1| + P_2|\psi_2\rangle\langle\psi_2|\right)\left(P_1|\psi_1\rangle\langle\psi_1| + P_2|\psi_2\rangle\langle\psi_2|\right)$$

$$= P_1^2|\psi_1\rangle\langle\psi_1| + P_2^2|\psi_2\rangle\langle\psi_2|$$

Since $P_1, P_2 \geq 0$ and $P_1 + P_2 = 1 \Rightarrow P_1^2 \leq P_1$; $P_2^2 \leq P_1$

Therefore: $\mathrm{Tr}\,\hat{\rho} - \mathrm{Tr}\,\hat{\rho}^2 = 1 - (P_1^2 + P_2^2) < 1$

for a pure state $\mathrm{Tr}\,\hat{\rho}_{|\psi\rangle} = \mathrm{Tr}\,\hat{\rho}^2_{|\psi\rangle} \Rightarrow \mathrm{Tr}\left(\hat{\rho}_{|\psi\rangle} - \hat{\rho}^2_{|\psi\rangle}\right) = 0$

It should be noticed that:

$$\hat{\rho}_{|\psi\rangle} = |\psi\rangle\langle\psi| = \left(\alpha|\psi_1\rangle + \beta|\psi_2\rangle\right)\left(\alpha^*\langle\psi_1| + \beta^*\langle\psi_2|\right)$$

$$= |\alpha|^2|\psi_1\rangle\langle\psi_2| + \alpha\beta^*|\psi_1\rangle\langle\psi_2| + \beta\alpha^*|\psi_2\rangle\langle\psi_1| + |\beta|^2|\psi_2\rangle\langle\psi_2|$$

$$\doteq \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{bmatrix}$$

in the $\{|\psi_1\rangle, |\psi_2\rangle\}$ basis

This is called in <u>classical optics</u> the <u>coherency</u> matrix

If we write:

$$\alpha = |\alpha| e^{ia}$$
$$\beta = |\beta| e^{ib}.$$

Then

$$\hat{\rho}_{|\psi\rangle} = \begin{bmatrix} |\alpha|^2 & |\alpha\beta| e^{i(a-b)} \\ |\alpha\beta| e^{-i(a-b)} & |\beta|^2 \end{bmatrix}$$

Suppose to have an ensemble of states $|\psi\rangle$ with random phases a and b. Moreover suppose that:

$$\overline{e^{i(a-b)}} \equiv C \ldots$$

then

$$\overline{\hat{\rho}_{|\psi\rangle}} = \begin{bmatrix} |\alpha|^2 & |\alpha\beta| C \\ |\alpha\beta| C^* & |\beta|^2 \end{bmatrix}$$

If $C = 0$ (total random) then

$$\overline{\hat{\rho}_{|\psi\rangle}} = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix} \equiv \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

If $C \neq 0$ we say that the state has still a partial coherence. Finally, notice that if

$$\hat{\rho} = P_1 |\psi_1\rangle\langle\psi_1| + P_2 |\psi_2\rangle\langle\psi_2|$$

and

$$|\psi_i\rangle = u_i |x\rangle + v_i |y\rangle \qquad |u_i|^2 + |v_i|^2 = 1$$

with

$$|\psi_1\rangle = \hat{U}|x\rangle$$

$$|\psi_2\rangle = \hat{U}|y\rangle$$

Then $$\hat{\rho} = P_1 \hat{U}|x\rangle\langle x|\hat{U}^+ + P_2 \hat{U}|y\rangle\langle y|\hat{U}^+$$
$$= \hat{U}\left(P_1 |x\rangle\langle x| + P_2 |y\rangle\langle y|\right)\hat{U}^+$$

Therefore, for a maximally-mixed state $P_1 = P_2 = \frac{1}{2}$

and $\hat{\rho} = \frac{1}{2}\hat{I}$ and $\hat{I}$ can be written in any basis.

$$\hat{I} = |x\rangle\langle x| + |y\rangle\langle y| = |R\rangle\langle R| + |L\rangle\langle L| = \hat{U}\left(|x\rangle\langle x| + |y\rangle\langle y|\right)\hat{U}^+ = \ldots$$

So, in this case if I have a first system made of two sources S_1 and S_2 emitting photons either x-pol. or y-pol. with 50% probability; and a second system with two sources S'_1 and S'_2 emitting photons either R-pol or L-pol. with 50% prob., I cannot distinguish the two systems.