Lecture 13

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IK-1, SS2017

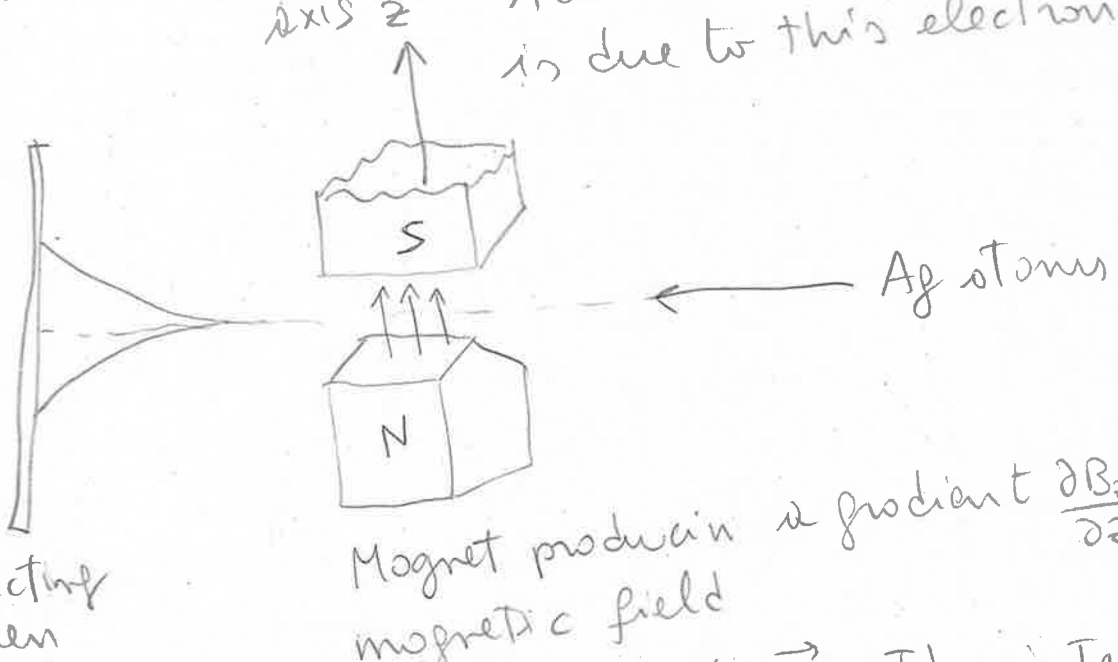
• SPIN ANGULAR MOMENTUM •

Silver (Ag) atoms = nucleus + 47 electrons

Stern-Gerlach experiment Frankfurt 1922

$= \underbrace{(N+46)}_{\text{behaves like a charged "sphere" with NO angular momentum}} + \underbrace{1 \text{ electron}}$

All the AM of the Ag atoms is due to this electron



Magnet producin a gradient $\frac{\partial B_z}{\partial z}$ of magnetic field

The electron has magnetic moment $\vec{\mu}$. The interaction energy is

$$U = -\vec{\mu}\cdot\vec{B}$$

and the force along the z axis:

$$F_z = \frac{\partial}{\partial z}(\vec{\mu}\cdot\vec{B}) \simeq \mu_z \frac{\partial B}{\partial z}$$

$$\mu_z \equiv \vec{\mu}\cdot\frac{\vec{B}}{|\vec{B}|} \simeq \vec{\mu}\cdot\vec{e}_z$$

Because $\vec{B}$ is almost parallel to the z-axis between the poles of the magnet.

In classical electrodynamics

$$\vec{\mu} = -\frac{e}{2mc}\vec{L}$$

$\vec{L}$ = ang. mom. of the electron

$-e$ = electronic charge $(e>0)$

m = electronic mass

c = speed of light

Therefore

$$\mu_z = -\beta_0 L_z$$

where

$$\beta_0 \equiv \frac{e\hbar}{2mc} \simeq 9.27401 \times 10^{-24} \frac{J}{Tesla}$$

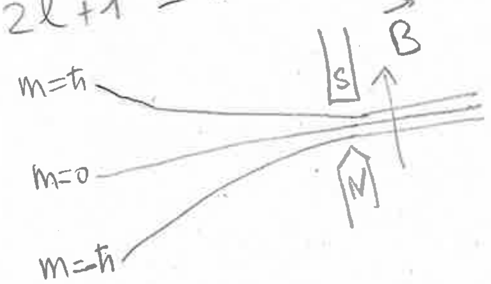
is the Bohr magneton

If $\vec{L}$ were ORBITAL angular momentum, then in quantum mechanics

$$L_z = m\hbar \quad \text{where} \quad m = -l, -l+1, \dots, l-1, l$$

with l integer. Therefore we would expect, in a SG experiment, $2l+1$ = odd number of traces:

example: $l = 1$



Instead SG found only 2 traces, implying:

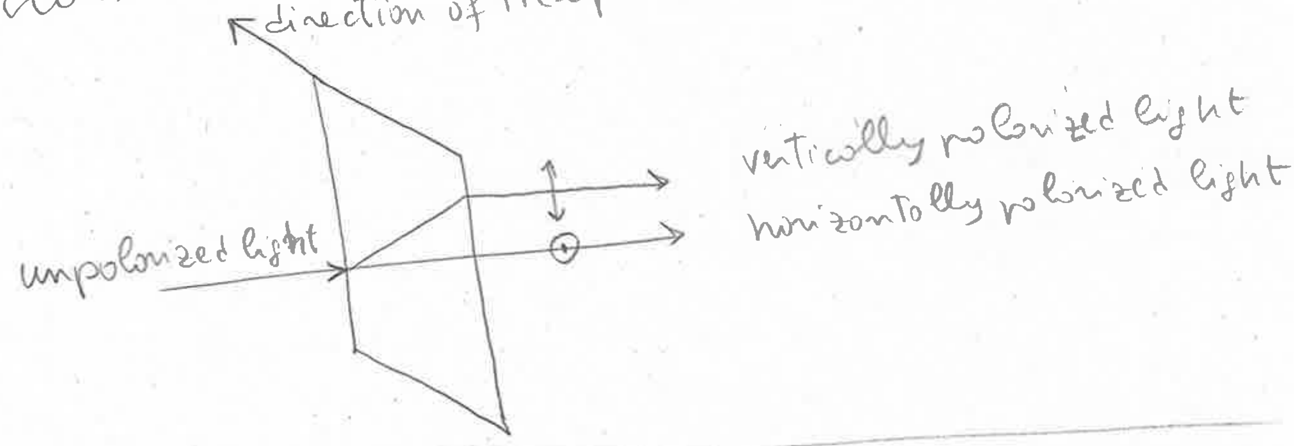
$$j = \frac{1}{2} \equiv s$$

$\downarrow$

the symbol "s" stands for SPIN

$$2s+1 = 2\cdot\frac{1}{2}+1 = 2 \quad \Rightarrow m = \left\{-\frac{1}{2}, -\frac{1}{2}+1 = \frac{1}{2}\right\}$$

Note: it should be noticed the analogy between the SG experiment and the transmission of unpolarized light across a calcite prism:



From lecture 14, we know that:

$$\langle j'm' | \hat{J}_\pm | j\,m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,\hbar\,\delta_{j,j'}\,\delta_{m\pm1,m'}$$

From now on, $j = j' = \frac{1}{2}$ and we use the symbols $\hat{S}_\pm, \hat{S}_x, \hat{S}_y, \hat{S}_z$ instead of $\hat{J}_\pm, \hat{J}_x, \hat{J}_y, \hat{J}_z$. Moreover

$$(j \mp m)(j \pm m + 1) = j^2 \cancel{\pm jm} + j \cancel{\mp jm} - m^2 \mp m$$

$$= j(j+1) - m(m \pm 1)$$

$$\rightarrow \frac{1}{2}\left(\frac{1}{2}+1\right) - m(m \pm 1) = \frac{3}{4} - m(m \pm 1) \qquad m = \pm\frac{1}{2}$$

$$m=+\frac{1}{2} \Rightarrow \frac{3}{4}-m(m\pm1)=\frac{3}{4}-\frac{1}{2}\left(\frac{1}{2}\pm1\right)=\begin{cases}0 & +\\ 1 & -\end{cases}$$

$$m=-\frac{1}{2} \Rightarrow \frac{3}{4}-m(m\pm1)=\frac{3}{4}+\frac{1}{2}\left(-\frac{1}{2}\pm1\right)=\begin{cases}1 & +\\ 0 & -\end{cases}$$

We use the shorthand notation

$$|j,m\rangle=|\tfrac{1}{2},m\rangle=\begin{cases}|+\rangle & \text{for } |\frac{1}{2},\frac{1}{2}\rangle\\ |-\rangle & \text{for } |\frac{1}{2},-\frac{1}{2}\rangle\end{cases}$$

Therefore

$$\hat{S}_+ \doteq \begin{bmatrix}\langle +|\hat{S}_+|+\rangle & \langle +|\hat{S}_+|-\rangle\\ \langle -|\hat{S}_+|+\rangle & \langle -|\hat{S}_+|-\rangle\end{bmatrix}=\hbar\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

$$\hat{S}_- \doteq \hbar\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

and

$$\hat{S}_x=\frac{1}{2}(\hat{S}_++\hat{S}_-)\doteq\frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$\hat{S}_y=\frac{1}{2i}(\hat{S}_+-\hat{S}_-)\doteq\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

$$\hat{S}_z \doteq \begin{bmatrix}\langle +|\hat{S}_z|+\rangle & \langle +|\hat{S}_z|-\rangle\\ \langle -|\hat{S}_z|+\rangle & \langle -|\hat{S}_z|-\rangle\end{bmatrix}=\frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

because $\hat{S}_z|\frac{1}{2},m\rangle=m\hbar|\frac{1}{2},m\rangle, \quad m=\pm\frac{1}{2}$

The three matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} ; \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are known as the <u>Pauli matrices</u> (PM)

In the context of spin theory, for historical reasons, the basis vectors $|+\rangle$ and $|-\rangle$ are denoted as χ_+ and χ_-, respectively:

$$|+\rangle \doteq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv \chi_+ \quad ; \quad |-\rangle \doteq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv \chi_-$$

$$\langle +| \doteq [1 \quad 0] \equiv \chi_+^\dagger \quad ; \quad \langle -| \doteq [0 \quad 1] \equiv \chi_-^\dagger$$

A generic ket $|\psi\rangle$ will be written as:

$$|\psi\rangle = |+\rangle\langle +|\psi\rangle + |-\rangle\langle -|\psi\rangle \doteq \begin{bmatrix} \langle +|\psi\rangle \\ \langle -|\psi\rangle \end{bmatrix}$$

What about spatial coordinates $|\underline{r}\rangle$?

The basis kets of a particle of spin $\frac{1}{2}$ can be described as vectors in the space given by the <u>tensor product</u> of the infinite-dimensional space generated by the position operator eigenvectors $\{|\underline{r}\rangle\}$, and the two-dimensional space generated by $\{|+\rangle, |-\rangle\}$. Therefore, we write:

$$|\underline{r}, \pm\rangle \equiv |\underline{r}\rangle \otimes |\pm\rangle \equiv |\underline{r}\rangle|\pm\rangle \leftarrow$$ equivalent writings, we can use any of them

Therefore using:

$$\hat{I} = \hat{I}_r \otimes \hat{I}_s = \int d^3r\, |\underline{r}\rangle\langle\underline{r}| \otimes \left(|+\rangle\langle+| + |-\rangle\langle-|\right)$$

Identity operator in ∞-d coordinate space $\mathcal{H}_r$

Identity operator in the 2-d spin space $\mathcal{H}_s$

$$= \int d^3r \left(|\underline{r},+\rangle\langle\underline{r},+| + |\underline{r},-\rangle\langle\underline{r},-|\right)$$

we can rewrite:

$$|\psi\rangle = \hat{I}|\psi\rangle = \int d^3r \left(|\underline{r},+\rangle\overbrace{\langle\underline{r},+|\psi\rangle}^{\equiv\psi_+(\underline{r})} + |\underline{r},-\rangle\overbrace{\langle\underline{r},-|\psi\rangle}^{\equiv\psi_-(\underline{r})}\right)$$

Then, the bra $\langle\underline{r}|$ applied to $|\psi\rangle$: $|\psi\rangle \to \langle\underline{r}|\psi\rangle$ must be seen as an application from $\mathcal{H} = \mathcal{H}_r \otimes \mathcal{H}_s$ to $\mathcal{H}_s$. Similarly the bras $\langle\pm|$ applied to $|\psi\rangle$ return a vector in $\mathcal{H}_r$:

$$\langle\underline{r}| : |\psi\rangle \to \langle\underline{r}|\psi\rangle \quad : \quad \mathcal{H}_r \otimes \mathcal{H}_s \to \mathcal{H}_s$$

$$\langle\pm| : |\psi\rangle \to \langle\pm|\psi\rangle \quad : \quad \mathcal{H}_r \otimes \mathcal{H}_s \to \mathcal{H}_r$$

So we can write:

$$\langle\underline{r}|\psi\rangle = \int d^3r' \left(\underbrace{\langle\underline{r}|\underline{r}',+\rangle}_{=\delta(\underline{r}-\underline{r}')|+\rangle}\psi_+(\underline{r}') + \underbrace{\langle\underline{r}|\underline{r}',-\rangle}_{=\delta(\underline{r}-\underline{r}')|-\rangle}\psi_-(\underline{r}')\right)$$

$$= |+\rangle\psi_+(\underline{r}) + |-\rangle\psi_-(\underline{r}) \doteq \begin{bmatrix}\psi_+(\underline{r})\\ \psi_-(\underline{r})\end{bmatrix}$$

– Properties of the Pauli Matrices (PM) –

The 2x2 Identity matrix I_2 is often denoted σ_0:

$$\sigma_0 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

and the Greek letters are used to write the four (4) Pauly matrices:

$$\{\sigma_\alpha\} \equiv \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$$

while Latin letter denotes

$$\{\sigma_i\} = \{\sigma_1, \sigma_2, \sigma_3\}$$

and the vector symbols $\underline{\sigma}$ or $\vec{\sigma}$ are shorthands for

$$\underline{\sigma} = \vec{\sigma} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}$$

1. Pauli Matrices (PM) are Hermitian:

$$\sigma_\alpha = \sigma_\alpha^\dagger$$

2. $$\sigma_\alpha^2 = \sigma_0 = I_2$$

3. From 2 and 1 it follows that:

$$\sigma_\alpha^2 = \sigma_\alpha \sigma_\alpha = \sigma_\alpha^\dagger \sigma_\alpha = \sigma_\alpha \sigma_\alpha^\dagger = I_2 \Rightarrow$$

$\Rightarrow$ PM are Unitary

4. Commutator

$$[\sigma_j, \sigma_k] = 2i\,\epsilon_{jkl}\,\sigma_l$$

5. Anticommutator

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk}$$

6. From 4 and 5 it follows that:

4 ⇒ $\sigma_j \sigma_k - \sigma_k \sigma_j = 2i\epsilon_{jkl}\sigma_l$

5 ⇒ $\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk}\sigma_0$

Summing 4 and 5 and dividing by 2:

$$\sigma_j \sigma_k = \sigma_0 \delta_{jk} + i\epsilon_{jkl}\sigma_l$$

7. $$\det(\sigma_i) = -1$$

8. $$\mathrm{Tr}(\sigma_i) = 0 \qquad \mathrm{Tr}(\sigma_0) = 2$$

9. From 6 and 8:

$$\mathrm{Tr}(\sigma_j^{\dagger}\sigma_k) = \mathrm{Tr}(\sigma_j\sigma_k)$$

$$= \delta_{jk}\underbrace{\mathrm{Tr}(\sigma_0)}_{=2} + i\epsilon_{jkl}\underbrace{\mathrm{Tr}(\sigma_l)}_{=0}$$

$$= 2\delta_{jk} \qquad \Leftrightarrow \text{orthonormality}$$

10. $$\frac{\sigma_0 + \sigma_3}{2} = \frac{1}{2}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\frac{\sigma_0 - \sigma_3}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{\sigma_1 + i\sigma_2}{2} = \frac{1}{2}\left\{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\right\} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

$$\frac{\sigma_1 - i\sigma_2}{2} = \frac{1}{2}\left\{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} - \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right\} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

11. Since any 2x2 matrix $A = [a_{ij}]$ can be written as:

$$A = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} = a_{11}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + a_{12}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + a_{21}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + a_{22}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

$$= a_{11}\left(\frac{\sigma_0 + \sigma_3}{2}\right) + a_{12}\left(\frac{\sigma_1 + i\sigma_2}{2}\right) + a_{21}\left(\frac{\sigma_1 - i\sigma_2}{2}\right) + a_{22}\left(\frac{\sigma_0 - \sigma_3}{2}\right)$$

$$= \sigma_0 \underbrace{\left(\frac{a_{11} + a_{22}}{2}\right)}_{= \frac{\mathrm{Tr}A}{2}} + \sigma_1\left(\frac{a_{21} + a_{12}}{2}\right) + \sigma_2\left(i\,\frac{a_{12} - a_{21}}{2}\right) + \sigma_3\left(\frac{a_{11} - a_{22}}{2}\right)$$

$$\equiv \sum_{\alpha=0}^{3} \sigma_\alpha a_\alpha = \begin{bmatrix}a_0 + a_3 & a_1 - ia_2\\ a_1 + ia_2 & a_0 - a_3\end{bmatrix}$$

where

$$\{a_\alpha\} = \frac{1}{2}\left\{a_{11} + a_{22},\ a_{12} + a_{21},\ i(a_{12} - a_{21}),\ a_{11} - a_{22}\right\}$$

From 9 it follows that:

$$\mathrm{Tr}(\sigma_\beta^{+} A) = \mathrm{Tr}(\sigma_\beta A) = \sum_{\alpha=0}^{3} a_\alpha \underbrace{\mathrm{Tr}(\sigma_\beta \sigma_\alpha)}_{2\delta_{\alpha\beta}} = 2a_\beta$$

So:

$$\boxed{a_\alpha = \frac{1}{2}\mathrm{Tr}(\sigma_\alpha^{+} A)}$$

Therefore, we say that the basis of PM

$$\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$$

is complete in the space $M^{2\times2}(\mathbb{C})$ of complex 2x2 matrices

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Homework: Demonstrate that the completeness of the PM can be expressed as:

$$\sum_{\alpha=0}^{3} [\sigma_\alpha]_{ab} [\sigma_\alpha^*]_{cd} = 2\delta_{ac}\delta_{bd}$$

with $a, b, c, d = 1, 2$

It should be noted that if $A = A^\dagger$, then

$a_{11}, a_{12} \in \mathbb{R}$ and $a_{21} = a_{12}^* \Rightarrow$

$\Rightarrow a_0, a_1, a_2, a_3 \in \mathbb{R}$

So, the PM give a basis for a real representation of Hermitian 2x2 matrices

12. If $\underline{u} = (u_1, u_2, u_3)$; $\underline{v} = (v_1, v_2, v_3)$, then

$$\underbrace{(\underline{\sigma}\cdot\underline{u})(\underline{\sigma}\cdot\underline{v})}_{\text{product between two matrices}} = (\underline{u}\cdot\underline{v})\sigma_0 + i\,\underbrace{\underline{\sigma}\cdot(\underline{u}\times\underline{v})}_{\text{Shorthand for the 2x2 matrix}}$$

$$\underline{\sigma}\cdot(\underline{u}\times\underline{v}) = \sigma_1(\underline{u}\times\underline{v})_1 + \sigma_2(\underline{u}\times\underline{v})_2 + \sigma_3(\underline{u}\times\underline{v})_3$$

– Spin and rotations –

In lecture 14 we learned that the operator $\hat{D}(\vec{n},\phi)$ rotating by an angle ϕ around the axis $\vec{n}$ (such that $\vec{n}\cdot\vec{n}=1$), the vector $|\psi\rangle \to |\psi_R\rangle = \hat{D}(\vec{n},\phi)|\psi\rangle$, was written as:

$$\hat{D}(\vec{n},\phi) = \exp\left(-\frac{i}{\hbar}\phi\,\vec{n}\cdot\hat{\underline{L}}\right)$$

where $\hat{\underline{L}} = (L_x, L_y, L_z)$ and $[\hat{L}_j, \hat{L}_k] = i\,\epsilon_{jkl}\hat{L}_l$.

Here the vector $|\psi\rangle$ was living in the infinite-dimensional vector space spanned by the eigenvectors $\{|\underline{r}\rangle\}$ of the position operator:

$$|\psi\rangle : \langle \underline{r}|\psi\rangle = \psi(\underline{r})$$

Now we want to show that if $|\alpha\rangle$ denotes a vector in the two-dimensional space spanned by $\{|+\rangle, |-\rangle\}$, then the rotation operator can be written as:

$$\hat{D}(\vec{n},\phi) = \exp\left(-\frac{i}{\hbar}\phi\,\vec{n}\cdot\hat{\underline{S}}\right)$$

and

$$|\alpha\rangle \xrightarrow{\hat{D}} |\alpha_R\rangle \equiv \hat{D}(\vec{n},\phi)|\alpha\rangle$$

To see this, let $\underline{r} = (x, y, z) \in \mathbb{R}^3$ a real-valued vector. We want to put $\underline{r}$ in correspondence to a ket $|\alpha\rangle = |+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle$, via the following
$\equiv |+\rangle\alpha_+ + |-\rangle\alpha_- \equiv |+\rangle a_1 e^{i\alpha_1} + |-\rangle a_2 e^{i\alpha_2}$

definition:

with: $(x, y, z) \propto (\langle\alpha|\hat{S}_x|\alpha\rangle, \langle\alpha|\hat{S}_y|\alpha\rangle, \langle\alpha|\hat{S}_z|\alpha\rangle)\frac{1}{\hbar}$

where
$$\frac{1}{\hbar}\langle\alpha|\hat{S}_x|\alpha\rangle = \frac{1}{2}\langle\alpha|\left(|+\rangle\langle-| + |-\rangle\langle+|\right)|\alpha\rangle$$
$$= \frac{1}{2}(\alpha_+^*\alpha_- + \alpha_-^*\alpha_+) = a_1 a_2 \cos(\alpha_2 - \alpha_1)$$
$$\frac{1}{\hbar}\langle\alpha|\hat{S}_y|\alpha\rangle = \frac{i}{2}\langle\alpha|\left(-|+\rangle\langle-| + |-\rangle\langle+|\right)|\alpha\rangle$$
$$= \frac{i}{2}(-\alpha_+^*\alpha_- + \alpha_-^*\alpha_+) = a_1 a_2 \sin(\alpha_2 - \alpha_1)$$
$$\frac{1}{\hbar}\langle\alpha|\hat{S}_z|\alpha\rangle = \frac{1}{2}\langle\alpha|\left(|+\rangle\langle+| - |-\rangle\langle-|\right)|\alpha\rangle$$
$$= \frac{1}{2}(|\alpha_+|^2 - |\alpha_-|^2) = \frac{1}{2}(a_1^2 - a_2^2)$$

If we write $\underline{r} = (x, y, z)$ in spherical coordinates, we find:
$$r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \left(a_1 a_2 \cos(\alpha_2 - \alpha_1), a_1 a_2 \sin(\alpha_2 - \alpha_1), \frac{a_1^2 - a_2^2}{2}\right)$$

If we identify $\phi = \alpha_2 - \alpha_1 + 2k\pi$

Then we have
$$\begin{cases} r\sin\theta = a_1 a_2 \\ r\cos\theta = \dfrac{a_1^2 - a_2^2}{2} \end{cases}$$

This implies:

$$r^2(\sin^2\theta + \cos^2\theta) = (a_1 a_2)^2 + \left(\frac{a_1^2 - a_2^2}{2}\right)^2$$

$$= \left(\frac{a_1^2 + a_2^2}{2}\right)^2 = \frac{1}{4}$$ because of normalization $\langle\alpha|\alpha\rangle = a_1^2 + a_2^2 = 1$

So $r = \frac{1}{2}$ and

$$\begin{cases} \sin\theta = 2a_1 a_2 \\ \cos\theta = a_1^2 - a_2^2 \end{cases}$$

whose solution is: $$\begin{cases} a_1 = \cos\left(\frac{\theta}{2}\right) \\ a_2 = \sin\left(\frac{\theta}{2}\right) \end{cases}$$

In summary, given the real vector

$$\underline{r} = (x, y, z) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

This is in correspondence, up to a multiplicative phase, with the ket

$$|\alpha\rangle = e^{i\alpha_1}\left(|+\rangle a_1 + |-\rangle a_2 e^{i(\alpha_2-\alpha_1)}\right)$$

where $\alpha_2 - \alpha_1 = \phi$, $a_1 = \cos\left(\frac{\theta}{2}\right)$ $a_2 = \sin\left(\frac{\theta}{2}\right)$

and

$$\boxed{\frac{\underline{r}}{|\underline{r}|} = \frac{2}{\hbar}\langle\alpha|(\hat{S}_x, \hat{S}_y, \hat{S}_z)|\alpha\rangle}$$

Now, we want to show that when $|\alpha\rangle \to |\alpha_R\rangle$ and

$$\frac{2}{\hbar}\langle\alpha|\hat{\underline{S}}|\alpha\rangle \to \frac{2}{\hbar}\langle\alpha_R|\hat{\underline{S}}|\alpha_R\rangle$$

$$= \frac{2}{\hbar}\langle\alpha|\hat{D}^{+}(\vec{n},\varphi)\hat{\underline{S}}\hat{D}^{+}(\vec{n},\varphi)|\alpha\rangle$$

then $\frac{\underline{r}}{|\underline{r}|} \to R(\vec{n},\varphi)\underline{r}\frac{1}{|\underline{r}|}$

where $R(\vec{n},\varphi)$ is the 3×3 real matrix associated with the rotation around $\vec{n}$ by an angle φ.

Equivalenty, we want to show that $\forall$ ket $|\alpha\rangle$:

$$\boxed{\langle\alpha|\hat{D}^{+}\hat{\underline{S}}\hat{D}|\alpha\rangle = R\langle\alpha|\hat{\underline{S}}|\alpha\rangle}$$

But since $|\alpha\rangle$ is arbitrary, then we want to show that:

$$\boxed{\hat{D}^{+}(\vec{n},\varphi)\hat{\underline{S}}\hat{D}(\vec{n},\varphi) = R(\vec{n},\varphi)\hat{\underline{S}}}$$

We demonstrate this in two different ways: a) algebraic way using the Backer-Hausdorff lemma and b) using The PM representation of $\hat{\underline{S}} \doteq \frac{\hbar}{2}\underline{\sigma}$

The BH lemma states that given the operators $\hat{A}$ and $\hat{B}$, then

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \cdots$$

Proof:

let $\hat{F}(\lambda) = e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}$ with $\lambda \in \mathbb{R}$

then

$$\hat{F}(\lambda) = \hat{F}(0) + \lambda \left.\frac{d\hat{F}}{d\lambda}\right|_{\lambda=0} + \frac{\lambda^2}{2!}\left.\frac{d^2F}{d\lambda^2}\right|_{\lambda=0} + \cdots$$

Clearly:

$$\hat{F}(0) = \hat{B}$$

then

$$\frac{d\hat{F}}{d\lambda} = \hat{A}\left(e^{\lambda\hat{A}}\hat{B}e^{-\lambda\hat{A}}\right) + e^{\lambda\hat{A}}\hat{B}\left(-\hat{A}e^{-\lambda\hat{A}}\right)$$

$$= \hat{A}\hat{F}(\lambda) - \hat{F}(\lambda)\hat{A} = [\hat{A},\hat{F}(\lambda)] \quad (1.15)$$

so that

$$\left.\frac{d\hat{F}}{d\lambda}\right|_{\lambda=0} = [\hat{A},\hat{B}]$$

Then, using (1.15)

$$\frac{d^2\hat{F}}{d\lambda^2} = \left[\hat{A},\frac{d\hat{F}(\lambda)}{d\lambda}\right] = [\hat{A},[\hat{A},\hat{F}(\lambda)]] \Rightarrow$$

$$\Rightarrow \left.\frac{d^2\hat{F}}{d\lambda^2}\right|_{\lambda=0} = [\hat{A},[\hat{A},\hat{B}]]$$

Then the result to prove follows by iteration

So:

$$\hat{D}^+(\vec{n},\varphi)\,\hat{S}_i\,\hat{D}(\vec{n},\varphi) =$$

$$= \exp\left(+\frac{i}{\hbar}\varphi\,\vec{n}\cdot\vec{S}\right)\hat{S}_i\,\exp\left(-\frac{i}{\hbar}\varphi\,\vec{n}\cdot\vec{S}\right)$$

$$\equiv e^{\hat{A}}\,\hat{B}\,e^{-\hat{A}}$$

where $\begin{cases} \hat{B} \equiv \hat{S}_i \\ \hat{A} \equiv \frac{i}{\hbar}\varphi\, n_j \hat{S}_j = \frac{i}{\hbar}\varphi\,\underline{n}\cdot\underline{\hat{S}} \end{cases}$

Then
$$[\hat{A},\hat{B}] = \frac{i}{\hbar}\varphi\, n_j \underbrace{[\hat{S}_j,\hat{S}_i]}_{=i\hbar\,\epsilon_{jik}\hat{S}_k}$$

$$= +\varphi\,\epsilon_{ijk} n_j \hat{S}_k = \varphi(\vec{n}\times\underline{\hat{S}})_i$$

and.
$$[\hat{A},[\hat{A},\hat{B}]] = \frac{i}{\hbar}\varphi\, n_\ell[\hat{S}_\ell,\ \varphi\,\epsilon_{ijk} n_j \hat{S}_k]$$

$$= \frac{i}{\hbar}\varphi^2 n_\ell\,\epsilon_{ijk} n_j \underbrace{[\hat{S}_\ell,\hat{S}_k]}_{=i\hbar\,\epsilon_{\ell k m}\hat{S}_m}$$

$$= -\varphi^2\,\epsilon_{ijk}\,\epsilon_{\ell k m}\, n_\ell n_j \hat{S}_m$$

$$= \varphi^2\,\epsilon_{ijk}\,\epsilon_{k\ell m}\, n_\ell n_j \hat{S}_m$$

$$= \varphi^2(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\, n_\ell n_j \hat{S}_m$$

$$= \varphi^2\Big(n_i\,\vec{n}\cdot\underline{\hat{S}} - \underbrace{\vec{n}\cdot\vec{n}}_{=1}\,\hat{S}_i\Big)$$

$$= -\varphi^2\hat{S}_i + \varphi^2 n_i(\vec{n}\cdot\underline{\hat{S}})$$

So:

$$e^{\hat{A}}\hat{S}_i e^{-\hat{A}} \simeq \hat{S}_i + \varphi(\vec{n}\times\hat{\underline{S}})_i - \frac{\varphi^2}{2}\hat{S}_i + \frac{\varphi^2}{2}n_i(\underline{n}\cdot\hat{\underline{S}}) +$$

$$+\frac{1}{3!}\frac{i}{\hbar}\varphi[\underline{n}\cdot\hat{\underline{S}}, -\varphi^2\hat{S}_i + \varphi^2\cancel{n_i(\vec{n}\cdot\hat{\underline{S}})}] + \dots$$

and: $$[\hat{A},[\hat{A},\hat{B}]] = -\varphi^2\hat{B} + \varphi^2 n_i \hat{A}\frac{\hbar}{i\varphi}$$

$$= -i\varphi\hbar\hat{A}n_i - \varphi^2\hat{B}$$

Therefore: $$\boxed{[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] = -\varphi^2[\hat{A},\hat{B}]}$$

so, at the Third iteration we start again.

This means:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] +$$

$$+\frac{1}{3!}(-\varphi^2[\hat{A},\hat{B}]) + \frac{1}{4!}\left(-\varphi^2[\hat{A},[\hat{A},\hat{B}]]\right) +$$

$$+\frac{1}{5!}\left(\varphi^4[\hat{A},\hat{B}]\right) + \frac{1}{6!}\left(\varphi^4[\hat{A},[\hat{A},\hat{B}]]\right) +$$

$$+\frac{1}{7!}\left(-\varphi^6[\hat{A},\hat{B}]\right) + \dots$$

$$= \underbrace{\hat{B}}_{=\hat{S}_i} + \underbrace{[\hat{A},\hat{B}]}_{=\varphi(\vec{n}\times\hat{\underline{S}})_i}\left(1 - \frac{\varphi^2}{3!} + \frac{\varphi^4}{5!} - \frac{\varphi^6}{7!} + \dots\right) + \underbrace{[\hat{A},[\hat{A},\hat{B}]]}_{=-\varphi^2\hat{S}_i + \varphi^2 n_i(\vec{n}\cdot\hat{\underline{S}})}\left(\frac{1}{2!} - \frac{\varphi^2}{4!} + \frac{\varphi^4}{6!} - \dots\right)$$

Since $$\sin\varphi \approx \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \dots$$

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \dots$$

$$1-\cos\varphi = \frac{\varphi^2}{2!} - \frac{\varphi^4}{4!} + \frac{\varphi^6}{6!} - \dots$$

Then:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{S}_i\cos\varphi + (\underline{n}\times\hat{\underline{S}})_i\sin\varphi + n_i(\underline{n}\cdot\hat{\underline{S}})(1-\cos\varphi)$$

but this is precisely the Rodrigues' rotation formula!

Therefore:

$$\boxed{\hat{D}^{+}(\vec{n},\varphi)\,\hat{\underline{S}}\,\hat{D}(\vec{n},\varphi) = \hat{\underline{S}}\cos\varphi + (\vec{n}\times\hat{\underline{S}})\sin\varphi + \vec{n}\,(\vec{n}\cdot\hat{\underline{S}})(1-\cos\varphi)}$$

c.v.d.

The demonstration using the PM representation of $\hat{\underline{S}}$ is much simpler. We begin with:

$$\exp\left(\frac{-i\,\hat{\underline{S}}\cdot\vec{n}\,\phi}{\hbar}\right) \doteq \exp\left(\frac{-i\,\underline{\sigma}\cdot\underline{n}\,\phi}{2}\right)$$

From the property 12 with $\underline{u} = \underline{v} = \underline{n} \in \mathbb{R}^3$, it follows that:

$$(\underline{\sigma}\cdot\underline{n})^2 = \sigma_0 \quad\Rightarrow\quad (\underline{\sigma}\cdot\underline{n})^k = \begin{cases} \sigma_0 & k \text{ even} \\ \underline{\sigma}\cdot\underline{n} & k \text{ odd} \end{cases}$$

Therefore:

$$\exp(-i\alpha\,\underline{\sigma}\cdot\underline{n}) = \sum_{k=0}^{\infty}\frac{(-i\alpha)^k}{k!}(\underline{\sigma}\cdot\underline{n})^k$$

$$\alpha \equiv \frac{\phi}{2} \qquad = \left(\sum_{k\,\mathrm{even}}\frac{(-i\alpha)^k}{k!}\right)\sigma_0 + \left(\sum_{k\,\mathrm{odd}}\frac{(-i\alpha)^k}{k!}\right)(\underline{\sigma}\cdot\underline{n})$$

$$= \sigma_0\cos\alpha - i(\underline{\sigma}\cdot\underline{n})\sin\alpha$$

Therefore:

$$\exp\left(\frac{-i\,\underline{\sigma}\cdot\underline{n}\,\phi}{2}\right) = \sigma_0\cos\left(\frac{\phi}{2}\right) - i(\underline{\sigma}\cdot\underline{n})\sin\left(\frac{\phi}{2}\right)$$

and

$$\exp\left(+\frac{i\underline{\sigma}\cdot\underline{n}\,\phi}{2}\right)\sigma_i\exp\left(-\frac{i\underline{\sigma}\cdot\underline{n}\,\phi}{2}\right) \equiv (0)$$

$$= \left[\sigma_0\cos\left(\frac{\phi}{2}\right) + i(\underline{\sigma}\cdot\underline{n})\sin\left(\frac{\phi}{2}\right)\right]\sigma_i\left[\sigma_0\cos\left(\frac{\phi}{2}\right) - i(\underline{\sigma}\cdot\underline{n})\sin\left(\frac{\phi}{2}\right)\right]$$

$$= \sigma_i\cos^2\left(\frac{\phi}{2}\right) - i\sigma_i\sigma_k n_k\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) + i\sigma_k n_k\sigma_i\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) +$$

$$+(\underline{\sigma}\cdot\underline{n})\sigma_i(\underline{\sigma}\cdot\underline{n})\sin^2\left(\frac{\phi}{2}\right)$$

$$= \sigma_i\cos^2\left(\frac{\phi}{2}\right) - i\underbrace{[\sigma_i,\sigma_k]}_{=2i\,\epsilon_{ikl}\sigma_l}n_k\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) + (\underline{\sigma}\cdot\underline{n})\sigma_i(\underline{\sigma}\cdot\underline{n})\sin^2\left(\frac{\phi}{2}\right)$$

$$= \sigma_i \cos^2\left(\frac{\phi}{2}\right) + 2\,\epsilon_{ikl} n_k \sigma_l \sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) + (\underline{\sigma}\cdot\underline{n})\,\sigma_i\,(\underline{\sigma}\cdot\underline{n})\,\sin^2\left(\frac{\phi}{2}\right)$$

where $(\underline{\sigma}\cdot\underline{n})\,\sigma_i\,(\underline{\sigma}\cdot\underline{n}) = (\underline{\sigma}\cdot\underline{n})\,(n_j \underbrace{\sigma_i\sigma_j}_{})$ ← use 6

$$= \sigma_0\,\delta_{ij} + i\,\epsilon_{ijk}\sigma_k n_j$$

$$= (\underline{\sigma}\cdot\underline{n})\,n_i + i\,\epsilon_{ijk}\,(n_l \underbrace{\sigma_l\sigma_k}_{= \sigma_0\delta_{lk} + i\epsilon_{lkm}\sigma_m})\,n_j$$

$$= (\underline{\sigma}\cdot\underline{n})\,n_i + \underbrace{i\,\epsilon_{ijl} n_j n_l}_{=0} + \underbrace{-\,\epsilon_{ijk}\,\epsilon_{lkm}\,n_l n_j \sigma_m}$$

$$= \epsilon_{ijk}\,\epsilon_{klm}\,n_l n_j \sigma_m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,n_l n_j \sigma_m$$

$$= n_i n_j \sigma_j - \underbrace{n_j n_j}_{=1}\sigma_i = n_i(\underline{n}\cdot\underline{\sigma}) - \sigma_i$$

$$= 2n_i(\underline{n}\cdot\underline{\sigma}) - \sigma_i$$

So:

$$(\circ) = \sigma_i \cos^2\left(\frac{\phi}{2}\right) + (\underline{n}\times\underline{\sigma})\sin\phi + 2n_i(\underline{n}\cdot\underline{\sigma})\underbrace{\sin^2\left(\frac{\phi}{2}\right)}_{=1-\cos\phi} - \sigma_i\sin^2\left(\frac{\phi}{2}\right)$$

$$\sigma_i \cos^2\left(\frac{\phi}{2}\right) - \sigma_i\sin^2\left(\frac{\phi}{2}\right) = \sigma_i$$

$$= \sigma_i\cos\phi + (\underline{n}\times\underline{\sigma})\sin\phi + n_i(\underline{n}\cdot\underline{\sigma})(1-\cos\phi)$$

in agreement with p. 18 c.v.d