LECTURE 20

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Consider a particle of spin $\frac{1}{2}$ (say, an electron), charge $-e<0$ and mass m, in a magnetic field $\vec{B}$.

The magnetic moment is $\mu = \frac{e}{mc}\frac{\hbar}{2}$ and the Hamiltonian is:

$$\hat{H} = \frac{e}{mc}\,\underline{\hat{S}}\cdot\underline{B}$$

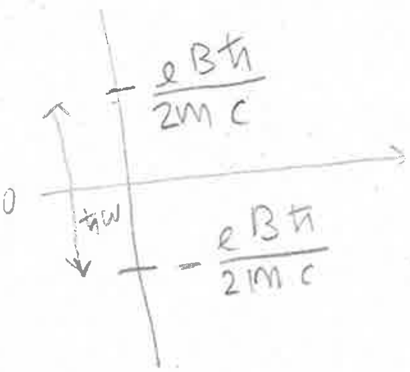
where $\underline{\hat{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ is the spin operator. Assume

$$\underline{B} = B\,\vec{e}_z \quad \text{with } B \geq 0 . \Rightarrow$$

$$\Rightarrow \quad \boxed{\hat{H} = \frac{eB}{mc}\hat{S}_z}$$

This implies that the eigenstates $|\pm\rangle$ of $\hat{S}_z$ are also eigenstate of $\hat{H}$ with energy

$$E_\pm = \pm \frac{eB\hbar}{2mc}$$



because

$$\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$$

Define:

$$\omega \equiv \frac{eB}{mc} \geq 0$$

and rewrite:

$$\boxed{\hat{H} = \omega\,\hat{S}_z}$$

The time-dependent Schrödinger equation is:

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \qquad 1.2)$$

If at time $t=0$ we have $|\psi(t=0)\rangle \equiv |\psi_0\rangle$, then at time $t>0$ the formal solution of 1.2 is:

$$|\psi(t)\rangle = \exp\left(-\frac{i\hat{H}t}{\hbar}\right)|\psi_0\rangle \equiv \hat{U}(t,0)|\psi_0\rangle$$

because

$$\frac{\partial}{\partial t}\left[\exp\left(\frac{-i\hat{H}t}{\hbar}\right)\right] = -\frac{i}{\hbar}\hat{H}\underbrace{\left[\exp\left(\frac{-i\hat{H}t}{\hbar}\right)\right]}_{\equiv \hat{U}(t,0)}$$

Since $\hat{H} = \hat{H}^{+}$, the operator $\hat{U}(t,0)$ is clearly unitary.

So

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = i\hbar\frac{\partial\hat{U}(t,0)}{\partial t}|\psi_0\rangle$$

$$= i\hbar\left(-\frac{i}{\hbar}\hat{H}\right)\underbrace{\hat{U}(t,0)|\psi_0\rangle}_{=|\psi(t)\rangle \text{ by def}}$$

$$= \hat{H}|\psi(t)\rangle$$

In our case $\hat{H} = \omega\hat{S}_z$

and

$$\boxed{\hat{U}(t,0) = \exp\left(-i\frac{\hat{S}_z}{\hbar}\omega t\right)}$$

If we remember the form of the rotation operator

$$\hat{D}(\vec{n},\phi) = \exp\left(-\frac{i}{\hbar}\vec{n}\cdot\hat{\vec{S}}\,\phi\right)$$

we see that

$$\hat{U}(t,0) = \hat{D}(\vec{e}_z, \omega t)$$

So, suppose we want to follow in time the behavior of the vector

$$\langle \psi(t) | \hat{\underline{S}} | \psi(t) \rangle =$$

$$= \langle \psi_0 | \hat{U}^\dagger(t,0) \hat{\underline{S}} \hat{U}(t,0) | \psi_0 \rangle$$

where
$$\begin{cases} \hat{S}_x = \frac{\hbar}{2} \left(|+\rangle\langle -| + |-\rangle\langle +| \right) \\ \hat{S}_y = \frac{i\hbar}{2} \left(-|+\rangle\langle -| + |-\rangle\langle +| \right) \\ \hat{S}_z = \frac{\hbar}{2} \left(|+\rangle\langle +| - |-\rangle\langle -| \right) \end{cases}$$

Then
$$\hat{U}^\dagger \hat{S}_x \hat{U} = \frac{\hbar}{2} e^{+i\hat{S}_z \frac{\omega t}{\hbar}} \left(|+\rangle\langle -| + |-\rangle\langle +| \right) e^{-i\hat{S}_z \frac{\omega t}{\hbar}}$$

$$= \frac{\hbar}{2} \left\{ e^{i\frac{\omega t}{2}} |+\rangle\langle -| e^{i\frac{\omega t}{2}} + e^{-i\omega t/2} |-\rangle\langle +| e^{-i\omega t/2} \right\}$$

because

$$\langle \pm | e^{-i\hat{S}_z \alpha} = \left(e^{i\hat{S}_z \alpha} |\pm\rangle \right)^* = \left(e^{i\left(\pm \frac{\hbar}{2}\right)\alpha} |\pm\rangle \right)^* = e^{\mp i \frac{\hbar \alpha}{2}} \langle \pm |$$

So
$$\hat{U}^\dagger \hat{S}_x \hat{U} = \frac{\hbar}{2} \left(e^{i\omega t} |+\rangle\langle -| + e^{-i\omega t} |-\rangle\langle +| \right)$$

Similarly

$$\hat{U}^\dagger \hat{S}_y \hat{U} = \frac{i\hbar}{2} e^{i\hat{S}_z \frac{\omega t}{\hbar}} \left(-|+\rangle\langle -| + |-\rangle\langle +| \right) e^{-i\hat{S}_z \frac{\omega t}{\hbar}}$$

$$= \frac{i\hbar}{2} \left(-e^{i\omega t} |+\rangle\langle -| + e^{-i\omega t} |-\rangle\langle +| \right)$$

Finally (and trivially):

$$\hat{U}^+ \hat{S}_z \hat{U} = \hat{S}_z$$

Since $e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t)$

it is convenient to rewrite

$$\hat{U}^+ \hat{S}_x \hat{U} = \Big[(\cos\omega t + i\sin\omega t)\,|+\rangle\langle -| + (\cos\omega t - i\sin\omega t)\,|-\rangle\langle +|\Big]\frac{\hbar}{2}$$

$$= \hat{S}_x \cos(\omega t) - \hat{S}_y \sin(\omega t)$$

$$\hat{U}^+ \hat{S}_y \hat{U} = \hat{S}_x \sin(\omega t) + \hat{S}_y \cos(\omega t)$$

The final result is therefore:

$$\langle \psi(t)|\hat{S}_x|\psi(t)\rangle = \langle \psi_0|\hat{S}_x|\psi_0\rangle \cos(\omega t) - \langle \psi_0|\hat{S}_x|\psi_0\rangle \sin(\omega t) \Leftrightarrow$$

$$\langle \hat{S}_x\rangle_t = \langle \hat{S}_x\rangle_0 \cos(\omega t) - \langle \hat{S}_y\rangle_0 \sin(\omega t)$$

$$\langle \psi(t)|\hat{S}_y|\psi(t)\rangle \equiv \langle \hat{S}_y\rangle_t = \langle \hat{S}_x\rangle_0 \sin(\omega t) + \langle \hat{S}_y\rangle_0 \cos\omega t$$

$$\langle \psi(t)|\hat{S}_z|\psi(t)\rangle \equiv \langle \hat{S}_z\rangle_t = \langle \hat{S}_z\rangle_0$$

This example shows the spin precession in a magnetic field. Notice that after a time $t = 2\pi/\omega$, the vector $\langle \hat{\underline{S}}\rangle_t$ comes back to its initial value. However, for the ket:

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\hat{S}_z \omega t\right)\left(|+\rangle\langle +|\psi_0\rangle + |-\rangle\langle -|\psi_0\rangle\right)$$

$$= e^{-i\omega t/2}\,|+\rangle\langle +|\psi_0\rangle + e^{+i\omega t/2}\,|-\rangle\langle -|\psi_0\rangle$$

So, it needs a time $t = \frac{4\pi}{\omega}$ to get back the original value

– New argument: sum of two angular momenta –

Consider two electrons, say 1 and 2, with spin operators $\hat{\underline{S}}_1 = (\hat{S}_{1x}, \hat{S}_{1y}, \hat{S}_{1z})$ and $\hat{\underline{S}}_2 = (\hat{S}_{2x}, \hat{S}_{2y}, \hat{S}_{2z})$.

By definition

$$[\hat{S}_{1i}, \hat{S}_{2j}] = 0$$

and

$$[\hat{S}_{ai}, \hat{S}_{ak}] = i\hbar\epsilon_{ikl}\hat{S}_{al} \qquad (a = 1, 2)$$

The operators $\hat{\underline{S}}_1$ act on a vector space V_1 of dimension 2 and $\hat{\underline{S}}_2$ on the space V_2.

The Total space

$$V = V_1 \otimes V_2$$

is the space spanned by:

$$|\tfrac{1}{2}, m_1\rangle \otimes |\tfrac{1}{2}, m_2\rangle \equiv |S_1, m_1, S_2, m_2\rangle$$

The basis $\{|S_1, m_1, S_2, m_2\rangle\}$ has four elements:

$$|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$$ (we omitted to write S_1 and S_2 for clarity)

We can define a <u>Total</u> spin operator

$$\hat{S}_j = \hat{S}_{1j} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2j}$$

where

$$[\hat{S}_j, \hat{S}_k] = (\hat{S}_{1j} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2j})(\hat{S}_{1k} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2k}) + $$
$$- (\hat{S}_{1k} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2k})(\hat{S}_{1j} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2j})$$
$$= \hat{S}_{1j}\hat{S}_{1k} \otimes \hat{I}_2 + \cancel{\hat{S}_{1j} \otimes \hat{S}_{2k}} + \cancel{\hat{S}_{1k} \otimes \hat{S}_{2j}} + \hat{I}_1 \otimes \hat{S}_{2j}\hat{S}_{2k} +$$
$$- \hat{S}_{1k}\hat{S}_{1j} \otimes \hat{I}_2 - \cancel{\hat{S}_{1k} \otimes \hat{S}_{2j}} - \cancel{\hat{S}_{1j} \otimes \hat{S}_{2k}} - \hat{I}_1 \otimes \hat{S}_{2k}\hat{S}_{2j}$$

$$= [\hat{S}_{1j}, \hat{S}_{1k}] \otimes \hat{I}_2 + \hat{I}_1 \otimes [\hat{S}_{2j}, \hat{S}_{2k}]$$

$$= i\hbar\, \epsilon_{jkl} \hat{S}_{1l} \otimes \hat{I}_2 + i\hbar\, \epsilon_{jkl} \hat{I}_1 \otimes \hat{S}_{2l}$$

$$= i\hbar\, \epsilon_{jkl} (\hat{S}_{1l} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2l}) = i\hbar\, \epsilon_{jkl} \hat{S}_l$$

So, the total spin has the same properties of each spin.

Therefore $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$

$$= (\hat{S}_{1x} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2x})^2 + (\hat{S}_{1y} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2y})^2 +$$

$$+ (\hat{S}_{1z} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{S}_{2z})^2$$

$$= \hat{S}_1^2 + \hat{S}_2^2 + 2 \hat{\underline{S}}_1 \cdot \hat{\underline{S}}_2$$

and this operator commutes with $\hat{S}_x, \hat{S}_y, \hat{S}_z$.

Moreover, it commutes with $\hat{S}_1^2$ and $\hat{S}_2^2$. To see this, it is enough to rewrite:

$$\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2 \hat{S}_{1z} \hat{S}_{2z} + \hat{S}_{1+} \hat{S}_{2-} + \hat{S}_{1-} \hat{S}_{2+}$$

and use the fact that: $[\hat{S}_2^2, \hat{S}_{2z}] = 0 = [\hat{S}_2^2, \hat{S}_{2\pm}]$.

This means that we have two different sets of commuting variables: (CSCO)

1) $\{\hat{S}_1^2, \hat{S}_{1z}, \hat{S}_2^2, \hat{S}_{2z}\}$ with eigenvalues $|S_1 m_1\rangle |S_2 m_2\rangle$

2) $\{\hat{S}^2, \hat{S}_z, \hat{S}_1^2, \hat{S}_2^2\}$ with eigenvalues denoted as:

$$|S, S_z, S_1, S_2\rangle$$

From the properties of AM algebra, we know that we must have

$$\hat{S}^2 |S, S_z, S_1, S_2\rangle = S(S+1)\hbar^2 |S, S_z, S_1, S_2\rangle$$

$$\hat{S}_z |S, S_z, S_1, S_2\rangle \doteq M\hbar |S, S_z, S_1, S_2\rangle$$

What are the values of S and M? What we have to do is to write the 4x4 matrices representing $\hat{S}^2$ and $\hat{S}_z$ in the tensor product basis $|S_1, m_1\rangle|S_2, m_2\rangle$ and diagonalize them

Note that by definition

$$[\hat{S}_z, \hat{S}_{1z}] = [\hat{S}_z, \hat{S}_{2z}] = 0 \qquad 1.7)$$

but, for example:

$$[\hat{S}^2, \hat{S}_{1z}] = 2i\hbar(-\hat{S}_{1y}\hat{S}_{2x} + \hat{S}_{1x}\hat{S}_{2y}) \neq 0$$

Homework: calculate explicitly $[\hat{S}^2, \hat{S}_{1z}]$

From 1.7) it follows that:

$$\hat{S}_z |S_1, m_1; S_2, m_2\rangle = (m_1 + m_2)\hbar |S_1, m_1; S_2, m_2\rangle$$

Therefor $|S_1, m_1; S_2, m_2\rangle$ is an eigenstate of $\hat{S}_z$ with eigenvalue $(m_1+m_2)\hbar$. Since $m_1, m_2 = \pm\frac{1}{2}$,

$$M = m_1 + m_2 = \begin{cases} -1 \\ 0 \\ 1 \end{cases}$$

$M = \pm 1$ is non degenerate because are given by $|++\rangle$ and $|--\rangle$ respectively.

However, $m_1+m_2=0$ is an eigenvalue of both $|+,-\rangle$ and $|-,+\rangle$ and their linear combinations.

So, eventually:

$$\frac{1}{\hbar}\hat{S}_z \doteq \begin{array}{c|cccc} & |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \\ \hline \langle ++| & 1 & 0 & 0 & 0 \\ \langle +-| & 0 & 0 & 0 & 0 \\ \langle -+| & 0 & 0 & 0 & 0 \\ \langle --| & 0 & 0 & 0 & -1 \end{array}$$

degenerate subspace.

Next, we calculate:

$$\hat{S}^2|++\rangle = 2\hbar^2|++\rangle$$
$$\hat{S}^2|+-\rangle = \hbar^2\left(|+-\rangle+|-+\rangle\right)$$
$$\hat{S}^2|-+\rangle = \hbar^2\left(|-+\rangle+|+-\rangle\right)$$
$$\hat{S}^2|--\rangle = 2\hbar^2|--\rangle$$

This implies that:

$$\frac{1}{\hbar^2}\hat{S}^2 \doteq \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

We must diagonalize the central matrix $S_0 \equiv \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

The eigenvalues are found by solving

$$0 = \det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2-1=0 \Rightarrow \begin{cases} \lambda=0 \\ \lambda=2 \end{cases}$$

So, $\hat{S}^2$ has only two distinct eigenvalues:

0 and $2\hbar^2$

The eigenvectors of So are:

$\frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$ for the eigenvalue 2

$\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$ for the eigenvalue 0

In summary: $\hat{S}^2$ has 2 distinct eigenvalues: 0 and 2, the latter being 3 times degenerate

Of course $S(S+1)=0 \Rightarrow S=0$ and $S(S+1)=2 \Rightarrow S=1$.

So, we denote:

$$|S, S_z, S_1, S_2\rangle\Big|_{\substack{S=0\\M=0}} \equiv |0,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$

← This state is called singlet

and the three degenerate states (called Triplet):

$$|1,1;1/2,1/2\rangle \equiv |1,1\rangle = |++\rangle$$

$$|1,0;1/2,1/2\rangle \equiv |1,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$

$$|1,-1;1/2,1/2\rangle \equiv |1,-1\rangle = |--\rangle$$

Homework: Show that the four states:

$$|0,0\rangle, |1,1\rangle, |1,0\rangle, |1,-1\rangle$$

form a complete and orthonormal basis

Note that the Triplet states are SYMMETRIC with respect to the exchange of the two particles, while the singlet is ANTISYMMETRIC

– General case –

What we have seen is just a particular case of the more general case of the sum of two angular momenta $\hat{\underline{J}}_1$ and $\hat{\underline{J}}_2$

$$\hat{J}_k = \hat{J}_{1k} \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{J}_{2k}$$

We give only the main results, without demonstrations. Who is interested in more details may consult vol II, chap. X of the Cohen-Tannoudji book.

So, we have 2 CSCO:

1) $\{\hat{J}_1^2, \hat{J}_{1z}, \hat{J}_2^2, \hat{J}_{2z}\} \xrightarrow{\text{eigenvectors}} |j_1, m_1, j_2, m_2\rangle$

2) $\{\hat{J}^2, \hat{J}_z, \hat{J}_1^2, \hat{J}_2^2\} \longrightarrow |J, M, j_1, j_2\rangle$

Where:
$$\hat{J}^2 |J, M, j_1, j_2\rangle = \hbar^2 J(J+1) |J, M, j_1, j_2\rangle$$

$$\hat{J}_z |J, M, j_1, j_2\rangle = \hbar M |J, M, j_1, j_2\rangle$$

and $$\hat{J}_a^2 |J, M, j_1, j_2\rangle = j_a(j_a+1)\hbar^2 |J, M, j_1, j_2\rangle \quad (a=1,2)$$

Both bases are orthonormal and complete and are related by a unitary transformation:

$$|J, M, j_1, j_2\rangle = \sum_{m_1}\sum_{m_2} |j_1, m_1, j_2, m_2\rangle \underbrace{\langle j_1, m_1, j_2, m_2 | J, M, j_1 j_2\rangle}$$

These objects are called Clebsh-Gordon coefficients!

These coefficients are zero unless

$$M = m_1 + m_2$$

because, by def:

$$\underbrace{(\hat{J}_z - \hat{J}_{1z} - \hat{J}_{2z})}_{=0 \text{ by def}} |J, M, j_1, j_2\rangle = 0$$

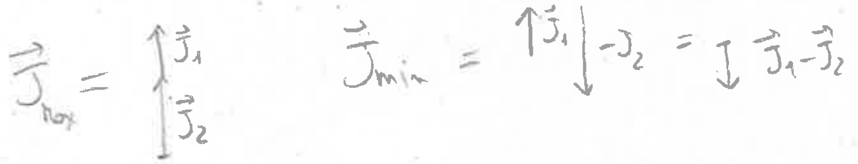
Multiplying from left by $\langle j_1 m_1, j_2 m_2|$ we get:

$$(M - m_1 - m_2)\langle j_1, m_1, j_2\, m_2 | J\, M j_1 j_2\rangle = 0$$

which proves our statement. Moreover CG coeff are non-zero only for:

$$|j_1 - j_2| \leq J \leq j_1 + j_2$$

This property is intuitive if you think of summing vectors:



To show this, first notice that in the basis $|j_1 m_1, j_2 m_2\rangle$, m_1 takes $2j_1+1$ values and m_2 takes $2j_2+1$ values.

So, there are

$$N = (2j_1+1)(2j_2+1)$$

different states with the same, fixed, values of j_1 e j_2.

This, of course, must be true also in the basis

$$|J M, j_1 j_2\rangle$$

therefore the pair of numbers (J, M) can take only N values.

Since $M = m_1 + m_2$, is clear that

$$\max(M) = \max(m_1) + \max(m_2)$$
$$= j_1 + j_2 = \max(J) \text{ by definition}$$

Then, all the other values of $M \geq 0$ will have the values:

$$M = j_1 + j_2 - k \geq 0, \quad \text{where} \quad k = 0, 1, 2, \ldots, k_{Max}$$

Let us determine k_{Max}. Suppose, without loss of generality, that $j_1 \geq j_2$. Then, all the values of m_1 and m_2 such that $M = m_1 + m_2 = j_1 + j_2 - k$ are:

$$\begin{array}{ccc} m_1 & m_2 & \\ & j_2 & 0 \\ j_1-k & j_2-1 & 1 \\ j_1-k+1 & j_2-2 & 2 \\ j_1-k+2 & \vdots & \vdots \\ \vdots & j_2-k & k \\ j_1 & & \end{array} \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} 2k+1 \text{ values}$$

$m_1 + m_2 = M$

Since $m_1 \geq -j_1$, then

$$j_1 - k \geq -j_1 \iff k \leq 2j_1$$

Similarly:

$$m_2 \geq -j_2 \Rightarrow j_2 - k \geq -j_2 \iff k \leq 2j_2$$

Since, by hynotesis, $j_1 \geq j_2$, it is enough to require

$$k \leq 2j_2 \leq 2j_1$$

Therefor

$$k = 0, 1, 2, \dots, 2j_2$$

that is

$$k_{max} = 2j_2$$

This means that

$$\min(M) = \min(j_1 + j_2 - k)$$

$$= j_1 + j_2 - k_{max}$$

$$= j_1 - j_2$$

If I started with $j_2 \geq j_1$ I would have found

$$\min(M) = j_2 - j_1$$

So, in general,

$$|j_1 - j_2| \leq M \leq j_1 + j_2$$

We have found all the nonnegative values of M.

However, in general,

$$M \in \{-J, -J+1, \dots, J-1, J\}$$

Therefore we can have states with:

$$M \in \{-|j_1-j_2|, \dots, |j_1-j_2|\} \qquad J = |j_1-j_2|$$

$$M \in \{-|j_1-j_2|-1, \dots, |j_1-j_2|+1\} \qquad J = |j_1-j_2|+1$$

$$\vdots$$

$$M \in \{-(j_1+j_2), \dots, (j_1+j_2)\} \qquad J = j_1+j_2$$

So, if again $j_1 \geq j_2$, the total number of these states is:

$$\sum_{J=j_1-j_2}^{j_1+j_2} (2J+1) = (2j_1+1)(2j_2+1) = N$$

in agreement with the previous result.

because:

$$Q = J - (j_1 - j_2) \Rightarrow$$

$$\Rightarrow \sum_{J=j_1-j_2}^{j_1+j_2} (2J+1) = \sum_{Q=0}^{2j_2} [2(Q+j_1-j_2)+1]$$

$$= 2\sum_{Q=0}^{2j_2} Q + \sum_{Q=0}^{2j_2} [2(j_1-j_2)+1]$$

$$= \cancel{2}\frac{2j_2(2j_2+1)}{\cancel{2}} + (2j_2+1)[2(j_1-j_2)+1]$$

$$= (2j_2+1)(2j_1+1)$$

The rotation operator is now:

$$\hat{D}^{(j_1)}(\vec{n},\phi)\otimes\hat{D}^{(j_2)}(\vec{n},\phi) = \exp\left(\frac{-i\hat{J}_1\cdot\vec{n}\,\phi}{\hbar}\right)\otimes\exp\left(\frac{-i\hat{J}_2\cdot\vec{n}\,\phi}{\hbar}\right) \equiv \hat{D}(\vec{n},\phi)$$

Note the same argument

For given values of j_1 and j_2, the representation of this operator in the base $|j_1 m_1\rangle\otimes|j_2 m_2\rangle$ is reducible;

$$\hat{D}(\vec{n},\phi) \doteq \begin{bmatrix} \hat{D}^{(j_1+j_2)} & & & \\ & \hat{D}^{(j_1+j_2-1)} & & \\ & & \ddots & \\ & & & D^{(j_1-j_2)} \end{bmatrix}$$

That is:

$$\hat{D}^{(j_1)}\otimes\hat{D}^{(j_2)} = \hat{D}^{(j_1+j_2)}\oplus\hat{D}^{(j_1+j_2-1)}\oplus\ldots\oplus D^{(|j_1-j_2|)}$$

This "block-reduction" occurs in the new basis $|J,M,j_1,j_2\rangle$.