Lecture 23

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– IDENTICAL PARTICLES –

Saying that a system is composed of N identical particles is equivalent to saying that there is no observable capable of distinguishing between them.

For example, consider two particles, say "a" and "b", of the same kind (e.g., two electrons, or two photons, etc).
Each of these particles possess a CSCO, namely a complete set of commuting observables $A, B, \ldots, Z$; such that the state $|a, b, .., z\rangle$ is uniquely determined.
For simplicity, let us denote with ξ the set of eigenvalues of $\hat{A}, \hat{B}, .., \hat{Z}$:

$$\xi = (a, b, \ldots, z)$$

and use the shorthand

$$|\xi\rangle = |a, b, .., z\rangle$$

The Hilbert space $\mathcal{H}$ of the system formed by the two particles is simply

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$$

and it is spanned by the basis vectors

$$\underbrace{|\xi_1, \xi_2\rangle_{ab} = |\xi_1\rangle_a |\xi_2\rangle_b = |\xi_1\rangle_a \otimes |\xi_2\rangle_b}_{\text{All equivalent ways to write the same vector}}$$

where from now on we adopt the convention that in the ket

$$|\xi_1, \xi_2\rangle_{ab}$$

The first label (here ξ_1) always refers to particle "a" and the second one (here ξ_2), to particle "b". Therefore, from now on we drop the labels "a" and "b".

Now, consider the two different states:

$$|\psi_{12}\rangle \equiv |\xi_1, \xi_2\rangle$$

$$|\psi_{21}\rangle \equiv |\xi_2, \xi_1\rangle$$

When the two-particle system is prepared in the state $|\psi_{12}\rangle$, particle a is characterized by the set of values ξ_1 and particle b by the values ξ_2. Vice versa for the system prepared in the state $|\psi_{21}\rangle$.

Suppose to make a measurement on the system and to find the values ξ_1 for one particle and ξ_2 for the other one. If the particles are indistinguishable, then we do not know if the state was $|\psi_{12}\rangle$ or $|\psi_{21}\rangle$ or even a linear combination

$$|\psi\rangle = c_1|\psi_{12}\rangle + c_2|\psi_{21}\rangle$$

of the two.

The degeneracy of $|\psi_{12}\rangle$ and $|\psi_{21}\rangle$ is called the "exchange degeneracy".

Let $\hat{P}$ be the permutation operator defined by:

$$\hat{P}|\xi_1, \xi_2\rangle = |\xi_2, \xi_1\rangle$$

or,

$$\hat{P}|\psi_{12}\rangle = |\psi_{21}\rangle$$

by definition, $\hat{P}^2 = \hat{I}$ ($\hat{X}_a$ and $\hat{X}_b$ are the same operator but acting on different particles)

If $\hat{A} \equiv \hat{X}_a \otimes \hat{I}_b$ and $\hat{B} \equiv \hat{I}_a \otimes \hat{X}_b$ such that:

1.3a) $$\hat{A}|\xi_1, \xi_2\rangle = (\hat{X}_a|\xi_1\rangle)|\xi_2\rangle = \xi_1|\xi_1, \xi_2\rangle$$

1.3b) $$\hat{B}|\xi_1, \xi_2\rangle = |\xi_1\rangle(\hat{X}_b|\xi_2\rangle) = \xi_2|\xi_1, \xi_2\rangle$$

Multiply 1.3a) by $\hat{P}$ from left:

$$\underbrace{\hat{P}(\hat{A}|\xi_1, \xi_2\rangle)} = \xi_1 \underbrace{\hat{P}|\xi_1, \xi_2\rangle}_{=|\xi_2, \xi_1\rangle}$$

$$= (\hat{P}\hat{A}\hat{P}^{-1})(\hat{P}|\xi_1, \xi_2\rangle)$$

$$= (\hat{P}\hat{A}\hat{P}^{-1})|\xi_2, \xi_1\rangle$$

$$\Leftrightarrow \quad (\hat{P}\hat{A}\hat{P}^{-1})|\xi_2, \xi_1\rangle = \xi_1|\xi_2, \xi_1\rangle \qquad 2.3)$$

Comparing 1.3b) with 2.3) we infer that:

$$\hat{P}\hat{A}\hat{P}^{-1} = \hat{B}$$

$$\Leftrightarrow \hat{P}(\hat{X}_a \otimes \hat{I}_b)\hat{P}^{-1} = \hat{I}_a \otimes \hat{X}_b$$

So, the operator $\hat{P}$ exchange the labels a and b.

* Example *

$$\hat{H} = \frac{\hat{\underline{P}}_1^2}{2m} + \frac{\hat{\underline{P}}_2^2}{2m} + V_{pair}(|\underline{r}_1 - \underline{r}_2|) + V(\underline{r}_1) + V(\underline{r}_2)$$

Clearly $\hat{H}(\hat{\underline{r}}_1, \hat{\underline{P}}_1; \hat{\underline{r}}_2, \hat{\underline{P}}_2) = \hat{H}(\hat{\underline{r}}_2, \hat{\underline{P}}_2; \hat{\underline{r}}_1, \hat{\underline{P}}_1)$

$\Rightarrow$ $[\hat{H}, \hat{P}] = 0 \Rightarrow \hat{P}$ is a constant of motion $\Leftrightarrow$

$\Leftrightarrow$ a state maintains its exchange symmetry upon time evolution because from $\hat{P}^2 = \hat{I}$ it follows that $\hat{P}^2 - \hat{I} = 0 \Leftrightarrow (\hat{P} - \hat{I})(\hat{P} + \hat{I}) = 0 \Rightarrow$

$\Rightarrow$ the eigenvalues of $\hat{P}$ are ± 1

Therefore, if $|\psi(t=0)\rangle$: $\hat{P}|\psi(t=0)\rangle = \pm|\psi(t=0)\rangle$, then this is true for any time $t > 0$.

More generally, in a system of N identical particle, any observable $\hat{A}$ must commute with $\hat{P}$.

$$[\hat{A}, \hat{P}] = 0$$

If this were not true, then we could distinguish the particles.

Consider now 3 particles "a", "b" and "c",

which
$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$$

and consider the normalized states

$$|\xi_i, \xi_j, \xi_k\rangle \underset{\text{short hand}}{\equiv} |i, j, k\rangle$$

with $i \neq j \neq k$ and suppose that each ξ_i can take only the three values

$$\xi_i = 1, 2, 3.$$

The 6 states we are considering are:

$$\begin{cases} |\Psi_S\rangle = \frac{1}{\sqrt{6}}\left[|123\rangle + |132\rangle + |231\rangle + |213\rangle + |312\rangle + |321\rangle\right] \\ |\Psi_A\rangle = \frac{1}{\sqrt{6}}\left[|123\rangle - |132\rangle + |231\rangle - |213\rangle + |312\rangle - |321\rangle\right] \end{cases}$$

$$\begin{cases} |\Psi_{M_1}\rangle = \frac{1}{2\sqrt{3}}\left[2|123\rangle - |132\rangle + 2|213\rangle - |231\rangle - |312\rangle - |321\rangle\right] \\ |\Psi_{M_2}\rangle = \frac{1}{2}\left[|132\rangle - |231\rangle + |312\rangle - |321\rangle\right] \end{cases}$$

$$\begin{cases} |\Psi_{M'_1}\rangle = \frac{1}{2\sqrt{3}}\left[2|123\rangle + |132\rangle - 2|213\rangle - |231\rangle - |312\rangle + |321\rangle\right] \\ |\Psi_{M'_2}\rangle = \frac{1}{2}\left[|132\rangle + |231\rangle - |312\rangle - |321\rangle\right] \end{cases}$$

We assume the normalization

$$\langle i\, j\, k \mid i'\, j'\, k' \rangle = \delta_{ii'}\, \delta_{jj'}\, \delta_{kk'}$$

Clearly

$$\hat{P}|\psi_S\rangle = |\psi_S\rangle$$

$$\hat{P}|\psi_A\rangle = \pm |\psi_A\rangle \; : \begin{cases} + & \text{even permutation} \\ - & \text{odd permutation} \end{cases}$$

There are $N! = 3!$ possible permutations in S_N (permutation group of N elements)

However, $\hat{P}|\psi_{M\alpha}\rangle \neq \lambda |\psi_{M\alpha}\rangle \qquad (\alpha = 1, 2)$

but $\hat{P}(c_1|\psi_{M_1}\rangle + c_2|\psi_{M_2}\rangle) = d_1|\psi_{M_1}\rangle + d_2|\psi_{M_2}\rangle$

This means that the set $\{|\psi_{M_1}\rangle, |\psi_{M_2}\rangle\}$ is <u>stable</u> against permutations.

The same is true for the set $\{|\psi_{M'_1}\rangle, |\psi_{M'_2}\rangle\}$

The states $|\psi_{M\alpha}\rangle$ and $|\psi_{M'\alpha}\rangle$ are said to have <u>Mixed</u> symmetry.

If $|\psi_{M_1}\rangle$ where physically realizable, the same <u>physical</u> state would be represented by the essentially diffent states $|\psi_{M_1}\rangle$ and $\hat{P}|\psi_{M_1}\rangle$

So, The Hilbert space $\mathcal{H}_3$ of a system of three identical particles can be decompose as the <u>direct sum</u> of four (4) subspaces:

$$\mathcal{H}_3 = \underset{\substack{\uparrow \\ \text{Totally symm.}}}{\mathcal{H}_S} \oplus \underset{\substack{\uparrow \\ \text{totally antsym}}}{\mathcal{H}_A} \oplus \underbrace{\mathcal{H}_M \oplus \mathcal{H}_{M'}}_{\equiv \mathcal{H}_\mathcal{M} \text{ mixde symm}}$$

the decomposition $\mathcal{H}_\mathcal{M} = \mathcal{H}_M \oplus \mathcal{H}_{M'}$ is <u>not</u> unique

Note: • From $[\hat{A}, \hat{P}] = 0$ for any obervable $\hat{A}$, it follows That the matrix elements between $\mathcal{H}_S$, $\mathcal{H}_A$ and $\mathcal{H}_\mathcal{M}$ are <u>zero</u>. This is called a <u>superselection</u> rule.

• Moreover, any eigenvalue of $\hat{A}$ in $\mathcal{H}_\mathcal{M}$ is at least doubly degenerate (exchange degeneracy)

• If $N=2$, we can olny have $\mathcal{H}_2 = \underbrace{\mathcal{H}_S}_{\text{3-dim}} \oplus \underbrace{\mathcal{H}_A}_{\text{1 dim}}$

• If $N>3$, in general we have:

$$\mathcal{H}_N = \mathcal{H}_S \oplus \mathcal{H}_A \oplus \mathcal{H}_{\mathcal{M}_1} \oplus \mathcal{H}_{\mathcal{M}_2} \oplus \cdots$$

Each $\mathcal{H}_{\mathcal{M}_i}$ has a complicated symmetry and furnish Higher-dimensional representations of S_N.

*A very important result is the <u>symmetrization Principle</u>:

- The pure states of a system of identical particles must be totally symmetric or antisymmetric under the exchange of any two of them.

Why? Well, if we admit that a physical system of N identical particles possesses some CSCO $\{A_1, A_2, \dots, A_r\}$, then its pure state cannot belong to any subspaces of mixed symmetry. In fact, if $a_1, \dots, a_r$ are the eigenvalues of $\hat{A}_1, \dots, \hat{A}_r$, then the state $|a_1, \dots, a_r\rangle$ is essentially unique. But if $|a_1, \dots, a_r\rangle \in \mathcal{H}_{mi}$, then we can find some permutation P such that

$$\hat{P}\,|a_1, \dots, a_r\rangle \neq e^{i\alpha}\,|a_1, \dots, a_r\rangle$$

for all α, and at the same time $\hat{P}|a_1, \dots, a_r\rangle$ is an eigenstate of $\hat{A}_1, \dots, \hat{A}_r$ with eigenvalues $a_1, \dots, a_r$, thus leading to a contradiction.

* In relativistic quantum theory is possible to establish the celebrated spin-statistic theorem (Pauli, 1941):

- The pure states of a system of identical particles are totally symmetric (antisymmetric) if their spin is an integer (half-odd integer).

Note: identical particles with totally symmetric wave functions obey Bose-Einstein statistics (these particles are called bosons); while particles with totally antisymmetric wave functions obey the Fermi-Dirac statistics (these particles are called Fermions).

Experimental data confirms that:

Fermions with spin $\frac{1}{2}$: electrons, nucleons, He^3, quarks, muon, tau + neutrinos

Bosons with spin 0 or 1: photons, He^4, mesons (π, K, J/ψ, ..)

For Fermions is valid the Pauli exclusion principle: "Two fermions cannot be in the same state".

Proof: if $\hat{P}_{ij}$ exchange ξ_i with ξ_j, then for a system of N Fermions:

$$P_{ij} \,|\xi_1, .., \xi_i, ..., \xi_j, .., \xi_N\rangle = -|\xi_1, .., \xi_j, .., \xi_i, .., \xi_N\rangle$$

$$\Rightarrow |\xi_1, .., \xi_i, .., \xi_i, .., \xi_N\rangle = 0 \text{ if } \xi_i = \xi_j.$$

Case study - Two-electron systems -

Two electrons, say "1" and "2":

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$\mathcal{H}_i$ is span by $\{|+\rangle_i, |-\rangle_i\}$ $(i=1,2)$

where $$\hat{S}_{i,z} |\pm\rangle_i = \pm\frac{\hbar}{2}|\pm\rangle_i$$

there are 4 independent product states

$$|+,+\rangle$$
$$|+,-\rangle$$
$$|-,+\rangle$$
$$|-,-\rangle$$

The states $|+,+\rangle$ and $|-,-\rangle$ are symmetric.

$$\hat{P}|\underline{+},\underline{+}\rangle = |\underline{+},\underline{+}\rangle$$

while $|+,-\rangle$ and $|-,+\rangle$ get mixed.

$$\hat{P}|+,-\rangle = |-,+\rangle$$
$$\hat{P}|-,+\rangle = |+,-\rangle$$

Then we can build the symmetric and antisymmetric linear combinations:

Sym: $\frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle)$

antisym: $\frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle)$

In summary, we can decompose $\mathcal{H}_1 \otimes \mathcal{H}_2$ as:

$$\mathcal{H}_1 \otimes \mathcal{H}_2 = \underbrace{\mathcal{H}_S}_{\substack{\text{3-dim} \\ \text{sym. space}}} \oplus \underbrace{\mathcal{H}_A}_{\text{1-dim antisym space.}}$$

$\mathcal{H}_S$ is span by the so-called Triplet: $\begin{cases} |+,+\rangle \\ \frac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle) \\ |-,-\rangle \end{cases}$

$\mathcal{H}_A$ is span by the singlet: $\frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle) \equiv |\psi_A\rangle$

If $\underline{r}_1, \underline{r}_2$ are the coordinates of the two electrons, the overall state $|\psi\rangle$ of the two electrons must be totally antisymmetric, with respect to the exchange of the two particle. This means that either:

$$\langle \underline{r}_1, \underline{r}_2 | \psi\rangle = \phi_A(\underline{r}_1, \underline{r}_2) \begin{cases} |++\rangle \\ \frac{1}{\sqrt{2}}(|+,-\rangle + |-+\rangle) \\ |--\rangle \end{cases} \qquad \phi_A(\underline{r}_1, \underline{r}_2) = -\phi_A(\underline{r}_2, \underline{r}_1)$$

or

$$\langle \underline{r}_1, \underline{r}_2 | \psi\rangle = \phi_S(\underline{r}_1, \underline{r}_2) \frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle) \qquad \phi_S(\underline{r}_1, \underline{r}_2) = \phi_S(\underline{r}_2, \underline{r}_1)$$

Any state of the form

$$|\psi_P\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$= (c_+|+\rangle + c_-|-\rangle) \otimes (d_+|+\rangle + d_-|-\rangle)$$

is said to be SEPARABLE or FACTORABLE.

Vice versa, a state of the form

$$|\psi_E\rangle = \sum_{i,j=1}^{2} c_{ij}\, |i,j\rangle \qquad \begin{pmatrix} 1 \Leftrightarrow + \\ 2 \Leftrightarrow - \end{pmatrix}$$

with $c_{ij} \neq a_i b_j$

is NON-separable or ENTANGLED.

This kind of state can always be written in a Schmidt form like:

$$|\psi_E\rangle = \sum_{i=1}^{2} \sqrt{\lambda_i}\, |u_i\rangle |v_i\rangle$$

where $\langle u_i|u_j\rangle = \delta_{ij} = \langle v_i|v_j\rangle$ and $\lambda_i \geq 0$

<u>Proof</u>: Consider the coefficients $C_{\alpha\beta}$ as the elements of a matrix

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

Then, make a <u>Singular Value Decomposition</u> of C:

$$C = UDV^\dagger$$

where U, V are unitary matrices and D is diagonal:

$$D_{ij} = \delta_{ij} \sqrt{\lambda_j}$$

Therefore,

$$C_{ij} = \sum_{k,\ell=1}^{2} U_{ik} D_{k\ell} V^{\dagger}_{\ell j}$$

$$= \sum_{k=1}^{2} \sqrt{\lambda_k}\, U_{ik} V^{*}_{jk} \quad \Rightarrow$$

$$\Rightarrow \quad |\psi_E\rangle = \sum_{i,j} \Big(\sum_k \sqrt{\lambda_k}\, U_{ik} V^{*}_{jk} \Big) |i\rangle |j\rangle$$

$$= \sum_k \sqrt{\lambda_k} \underbrace{\Big(\sum_i U_{ik} |i\rangle \Big)}_{\equiv |u_k\rangle} \underbrace{\Big(\sum_j V^{*}_{jk} |j\rangle \Big)}_{\equiv |v_k\rangle}$$

$$= \sum_k \sqrt{\lambda_k}\, |u_k, v_k\rangle$$

By def:

$$\langle u_i | u_j \rangle = \sum_{k,\ell} \big(\langle k | U^{*}_{ki} \big) \underbrace{\big(U_{\ell j} | \ell \rangle \big)}_{\delta_{k\ell}}$$

$$= \sum_k U^{*}_{ki} U_{kj}$$

$$= \sum_k U^{\dagger}_{ik} U_{kj}$$

$$= (U^{\dagger} U)_{ij} = \delta_{ij} \quad \text{c.v.d} \quad (\text{same proof for } \langle v_i | v_j \rangle)$$

This is called a Schmidt decomposition.
The generalization to N dimension is Trivial.
It is valid only for pure states.

The quantity

$$K = \frac{1}{\sum_{i=1}^{2} \lambda_i^2}$$

is called the Schmidt rank of the state and gives a measure of entanglement. For a factorable state as $|+,+\rangle = |1,1\rangle$

we have $\lambda_1 = 1$, $\lambda_2 = 0 \Rightarrow K = 1$

For a singlet state:

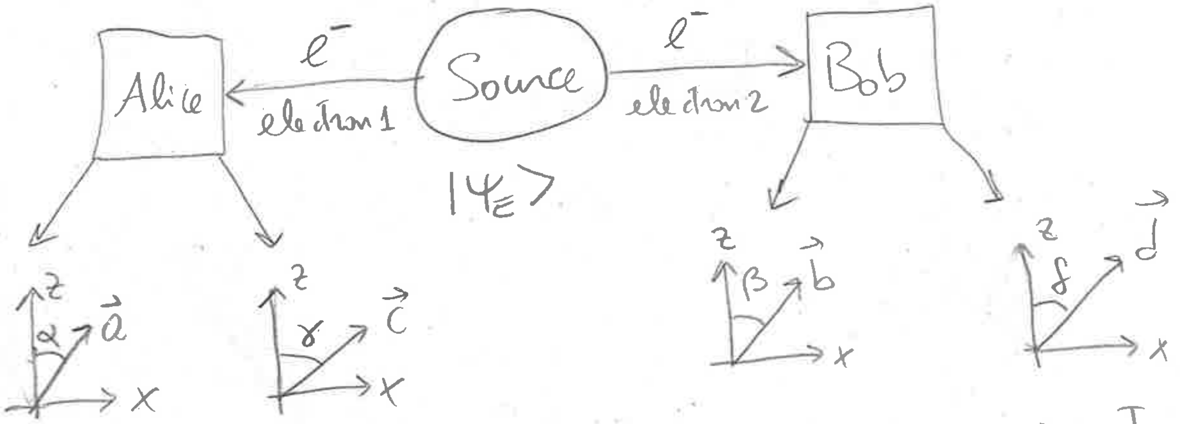
$$\frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right) = \frac{1}{\sqrt{2}}|u_1\rangle|v_1\rangle + \frac{1}{\sqrt{2}}|u_2\rangle|v_2\rangle$$

where $\{|u_1\rangle, |u_2\rangle\} = \{|+\rangle, |-\rangle\}$

$\{|v_1\rangle, |v_2\rangle\} = \{|+\rangle, -|-\rangle\}$

and $\sqrt{\lambda_1} = \sqrt{\lambda_2} = \frac{1}{\sqrt{2}} \Rightarrow K = 2$ (Maximally entangled)

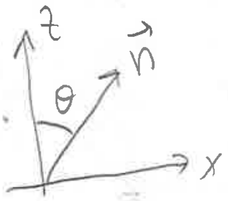
– Bell's inequality –



Alice and Bob have two Stern-Gerlach (SG) setups. They can choose the orientation of the magnetic field parallel to either $\vec{a}$ or $\vec{c}$ (Alice) and parallel to either $\vec{b}$ or $\vec{d}$ (Bob).

The source emits pairs of electrons in the entangled state
$$|\psi_E\rangle = \frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right)$$
(z is reference quantization axis for the source)

From now on we choose the units:
$$\hbar = 1$$
and we recall that if we define $\hat{S}_{\underline{n}} \equiv \hat{S}\cdot\vec{n}$ with $\vec{n}$ such that:



$$\vec{n} = \sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_z$$

Then
$$\hat{S}_{\underline{n}}|\vec{n},\pm\rangle = \pm\frac{1}{2}|\vec{n},\pm\rangle$$

where

$$\begin{cases} |\vec{n},+\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle \\ |\vec{n},-\rangle = -\sin\frac{\theta}{2}|+\rangle + \cos\frac{\theta}{2}|-\rangle \end{cases}$$

Let $P_+(\alpha)$ the probability that Alice detect an electron in the state $|\vec{a},+\rangle$ irrespective of Bob measurement is calculated as, for example,

$$P_+(\alpha) = \left|\left(|\vec{a},+\rangle|+\rangle\right)^+ |\psi_E\rangle\right|^2 + \left|\left(|\vec{a},+\rangle|-\rangle\right)^+ |\psi_E\rangle\right|^2$$

$$= |\langle\vec{a},+;+|\psi_E\rangle|^2 + |\langle\vec{a},+;-|\psi_E\rangle|^2$$

where

$$\langle\psi_E|\vec{a},+;-\rangle = \frac{1}{\sqrt{2}}\left(\langle+,-| - \langle-,+|\right)\left(\cos\frac{\alpha}{2}|+\rangle + \sin\frac{\alpha}{2}|-\rangle\right)|-\rangle$$

$$= \frac{1}{\sqrt{2}}\cos\frac{\alpha}{2}$$

$$\langle\psi_E|\vec{a},+;+\rangle = -\frac{1}{\sqrt{2}}\sin\frac{\alpha}{2}$$

and $P_+(\alpha) = \frac{1}{2}\cos^2\frac{\alpha}{2} + \frac{1}{2}\sin^2\frac{\alpha}{2} = \frac{1}{2}$

The joint probability $P_{++}(\alpha,\beta)$ to detect the electrons in the state $|\vec{a},+\rangle|\vec{b},+\rangle$ is:

$$P_{++}(\alpha,\beta) = \left|\left(\langle\vec{a},+|\langle\vec{b},+|\right)\frac{1}{\sqrt{2}}\left(|+,-\rangle - |-,+\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\left(\cos\frac{\alpha}{2}\sin\frac{\beta}{2} - \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\right)\right|^2 = \frac{1}{2}\sin^2\left(\frac{\alpha-\beta}{2}\right)$$

Similarly, the probability

$$P_{+-}(\alpha,\beta)$$

to find the electrons in the state

$$|\vec{a},+\rangle|\vec{b},-\rangle$$

is

$$P_{+-}(\alpha,\beta) = \frac{1}{2}\left|\langle\vec{a},+|+\rangle\langle\vec{b},-|-\rangle - \langle\vec{a},+|-\rangle\langle\vec{b},-|+\rangle\right|^2$$

$$= \frac{1}{2}\left|\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\left(-\sin\frac{\beta}{2}\right)\right|^2$$

$$= \frac{1}{2}\left|\cos\left(\frac{\alpha-\beta}{2}\right)\right|^2 = \frac{1}{2}\cos^2\left(\frac{\alpha-\beta}{2}\right).$$

Finally, the conditional probability

$$P_c(\alpha|\beta)$$

that Alice finds her electrons either in $|\vec{a},+\rangle$ or $|\vec{a},-\rangle$ given that Bob has detected his electron in either $|\vec{b},+\rangle$ or $|\vec{b},-\rangle$, is equal to:

$$P(\vec{a},+;\vec{b},+) + P(\vec{a},-;\vec{b}+) = P(\vec{a},+;\vec{b},-) + P(\vec{a},+;\vec{b},-)$$

$$= \sin^2\left(\frac{\alpha-\beta}{2}\right)$$

The outcome of the measurement on electron 1 appears to be influenced by the orientation β of Bob's setup, even though the two electrons may be well separated at the time of measurement. This is the so-called <u>non-locality paradox.</u>

* Bell's inequality *

a, b, c, d = dichotomic variables
they take only 2 values: ± 1

Then $(a+c)b-(a-c)d=\pm 2$

-Proof-

We can have either $a+c=0 \Rightarrow c=-a \Rightarrow a-c=2a$

or $a-c=0 \Rightarrow c=a \Rightarrow a+c=2a$

In the first case:

$$\left[(a+c)b-(a-c)d\right]_{c=-a} = -2ad = \pm 2$$

In the second case

$$\left[(a+c)b-(a-c)d\right]_{c=a} = 2ab = \pm 2$$

-end of the proof-

Alice can choose between two orientations α and γ.

Bob can choose between β and δ.

If Alice finds the electron in the state $|\vec{\alpha}, \pm\rangle$, she assignes to the random variable a the value ± 1, and so on. In each run of the experiment, the 4 variables a, b, c, d are constrained by:

$$(a+c)b-(a-c)d = \pm 2$$

$$\Leftrightarrow \quad ab+cb+cd-ad = \pm 2$$

We repeat the measurement N times:

1st run: $a_1 b_1 + c_1 b_1 + c_1 d_1 - a_1 d_1 = s_1 2$ $\qquad s_j = \pm 1$ (sign)

2nd run: $a_2 b_2 + c_2 b_2 + c_2 d_2 - a_2 d_2 = s_2 2$

$\vdots$

jth run: $a_j b_j + c_j b_j + c_j d_j - a_j d_j = s_j 2$

$\vdots$

Nth run: $a_N b_N + c_N b_N + c_N d_N - a_N d_N = s_N 2$ $\quad \leftarrow$ Take the sum

$$\sum_{j=1}^{N} (a_j b_j + c_j b_j + c_j d_j - a_j d_j) = 2 \sum_{j=1}^{N} s_j$$

where $\quad -N \leq \sum_{j=1}^{N} s_j \leq N$ by def.

If we define:

$$\langle ab \rangle \equiv \frac{1}{N} \sum_{i=1}^{N} a_i b_i$$

Then:

$$\boxed{|\langle ab \rangle + \langle cb \rangle + \langle cd \rangle - \langle ad \rangle| \leq 2}$$

This is called the CHSH inequality.

let us calculate the CORRELATIONS $\langle ab \rangle$:

a_i	b_i	$a_i b_i$	number of occurrences
1	1	1	N_{++}
1	-1	-1	N_{+-}
-1	1	-1	N_{-+}
-1	-1	1	N_{--}

by def:

$$N_{++} + N_{+-} + N_{-+} + N_{--} = N$$

Therefore:

$$\langle ab \rangle = \frac{1}{N} \sum_{i=1}^{N} a_i b_i$$

$$= \frac{1}{N} \left(N_{++}(1) + N_{+-}(-1) + N_{-+}(-1) + N_{--}(1) \right)$$

$$= \frac{N_{++} - N_{+-} - N_{-+} + N_{--}}{N_{++} + N_{+-} + N_{-+} + N_{--}}$$

or, equivalently,

$$\langle ab\rangle = \frac{N_{++}}{N} - \frac{N_{+-}}{N} - \frac{N_{-+}}{N} + \frac{N_{--}}{N}$$

For $N \gg 1$, according to the frequency definition of probability:

$$\frac{N_{\alpha\beta}}{N} \simeq P(\vec{a},\alpha;\vec{b},\beta) \qquad (\alpha = \pm)$$

Therefore, Quantum Mechanics predicts:

$$\langle ab\rangle = P(\vec{a},+;\vec{b}+) - P(\vec{a},+;\vec{b}-) - P(\vec{a},-;\vec{b},+) + P(\vec{a},-;\vec{b},-)$$

$$= \frac{1}{2}\sin^2\left(\frac{\alpha-\beta}{2}\right) - \frac{1}{2}\cos^2\left(\frac{\alpha-\beta}{2}\right) - \frac{1}{2}\cos^2\left(\frac{\alpha-\beta}{2}\right) + \frac{1}{2}\sin^2\left(\frac{\alpha-\beta}{2}\right)$$

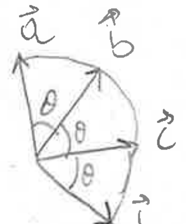
$$= \sin^2\left(\frac{\alpha-\beta}{2}\right) - \cos^2\left(\frac{\alpha-\beta}{2}\right)$$

$$= -\cos(\alpha-\beta)$$

and the CHSH inequality gives:

$$\boxed{|\cos(\alpha-\beta) + \cos(\gamma-\beta) + \cos(\gamma-\delta) - \cos(\alpha-\delta)| \leq 2}$$

However, if you choose:



$\theta = \pi/4$

Then:

$$\left.\begin{array}{l}\alpha-\beta=\frac{\pi}{4} \Rightarrow \cos(\alpha-\beta)=\frac{1}{\sqrt{2}}\\[2ex] \gamma-\beta=\frac{\pi}{4} \Rightarrow \cos(\gamma-\beta)=\frac{1}{\sqrt{2}}\\[2ex] \gamma-\delta=\frac{\pi}{4} \Rightarrow \cos(\gamma-\delta)=\frac{1}{\sqrt{2}}\\[2ex] \alpha-\delta=\frac{3\pi}{4} \Rightarrow -\cos(\alpha-\delta)=+\frac{1}{\sqrt{2}}\end{array}\right\} \Rightarrow$$

$\Rightarrow CHSH = \left|4\cdot\frac{1}{\sqrt{2}}\right| = 2\sqrt{2} > 2$! contradiction!

Where is the problem? The problem is in the COUNTERFACTUAL assumptions, because either we choose to measure

$$\vec{a},\vec{b} \quad \text{OR} \quad \vec{c},\vec{d}$$

but NOT:

$$\vec{a},\vec{b} \quad \text{AND} \quad \vec{c},\vec{d}$$

at the same time.