Lecture 3

– Probability –

S is a <u>set</u>, called SAMPLE SPACE

example $S=\{1,2,3,4,5,6\}$ ← output of rolling a dice

A, B, ... possible subsets of S

The probability (Kolmogorov) P is ~~a~~ a real-valued function such that:

1. $\forall A \in S, \; P(A) \geq 0$

2. if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$

3. $P(S) = 1$

– example – for the dice $S=\{n\}$, ~~n~~ $=1,..,6$

Then $P(k) = \frac{1}{6}$ $\quad k=1,..,6$

$$\sum_{k=1}^{6} P(k) = 1$$

for example $A=\{1,6\}$; $B=\{3,5\}$

Then $A \cap B = 0$ and $P(A \cup B) = P(A) + P(B)$

$= [P(1)+P(6)] + [P(3)+P(5)]$

$= \frac{4}{6} = \frac{2}{3}$

here $P(A)$ = prob to get <u>either</u> 1 or 6

$P(A \cup B)$ = prob to get <u>either</u> 1, or 6, or 3, or 5

P2)

If $A \cap B \neq \emptyset$ Then

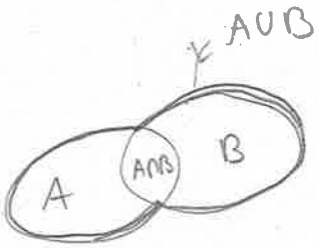
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$A = \{1, 6\}$; $B = \{3, 6\}$

$A \cap B = 6$

$$P(A \cap B) = [P(1) + P(6)] + [P(3) + \underbrace{\cancel{P(6)}] - \cancel{P(6)}}_{\text{so IT is not conted Twice}}$$

$$= \frac{3}{6} = \frac{1}{2}$$



Proof:

(* if $A \cap B \neq \emptyset$

$$A \cup B = A \cup (B - A \cap B) \Rightarrow A \cap (B - A \cap B) = 0$$

$$B = (B - A \cap B) \cup (A \cap B) \Rightarrow (B - A \cap B) \cap (A \cap B) = 0$$

$$P(A \cup B) = P(A) + P(B - A \cap B)$$

$$P(B) = P(B - A \cap B) + P(A \cap B)$$

Take the difference $\Rightarrow$

$$P(A \cup B) - P(B) = P(A) + P(A \cap B)$$

$$\Rightarrow \boxed{P(A \cup B) = P(A) + P(B) + P(A \cap B)}$$

*)

– conditional probability $P(A|B)$ (read as P of A given B)

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad \text{if } P(B)>0$$

Since $A\cap B = B\cap A$ we can exchange A and B to obtain

$$P(B|A) = \frac{P(A\cap B)}{P(A)} \; ; \quad P(A)>0$$

use above
$$= P(A|B)\,\frac{P(B)}{P(A)}$$

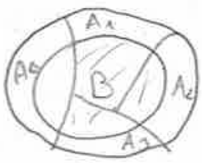
and
$$\boxed{P(A|B) = \frac{P(A)}{P(B)}\,P(B|A)}$$
Bayes' theorem

If $A_1, A_2, \ldots$ are disjoint subsets: $\cap_i A_i = \phi$, $\cup_i A_i = S$

then
$$P(B\cap A_i) = P(B|A_i)\,P(A_i)$$

if I sum over i:
$$\boxed{\sum_i P(B\cap A_i) = P(B) = \sum_i P(B|A_i)\,P(A_i)}$$

This is called the LAW of TOTAL PROBABILITY



From above we have

$$P(A\cap B) = P(A|B)\,P(B)$$

$$P(A\cap B) = P(B|A)\,P(A)$$

But if the event are INDEPENDENT:

$$\left.\begin{array}{l} P(A|B) = P(A) \\ P(B|A) = P(B) \end{array}\right\} \Rightarrow P(A\cap B) = P(A)\cdot P(B)$$

-Example- Two dices:

1) $A=1$, $B=2$

$$P(A\cap B) = P(A)\cdot P(B) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$$

$$P(A|B) = P(1|2) = \frac{P(A\cap B)}{P(B)} = \frac{\cancel{\frac{1}{6}}\cdot\frac{1}{6}}{\cancel{\frac{1}{6}}} = \frac{1}{6}$$

2) $A=\{1,6\}$, $B=\{3,5\}$

$P(A|B) = P$ to obtain 1 or 6 in dice #1 provided that dice #2 got 3 or 5

$$P(A\cap B) = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9} \quad ; \quad P(B) = \frac{2}{6} = \frac{1}{3}$$

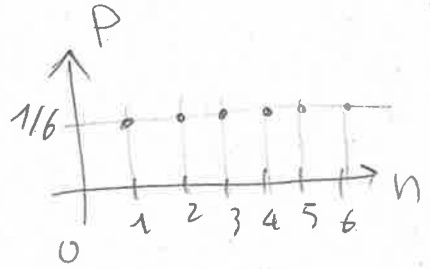
$$P(A|B) = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3} = P(A) \quad \underline{\text{c.v.d}}$$

— RANDOM VARIABLES —

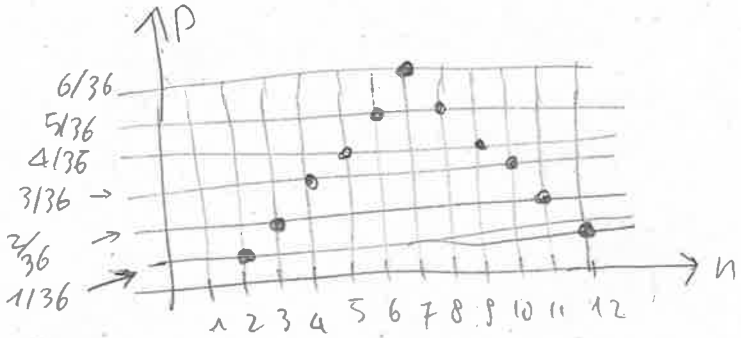
"A random variable is a number assigned to an element of the sample space S"

— example — 1 dice

$n = 1, 2, \dots, 6$



2 dices: $n = n_1 + n_2$
$= 2, 3, \dots, 12$



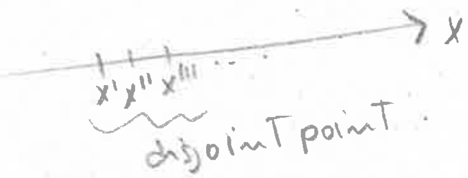
$$P(7) = P(4 \cap 3) + P(5 \cap 2) + P(6 \cap 1) + P(3 \cap 4) + P(2 \cap 5) + P(1 \cap 6) = \frac{6}{36}$$

In physics deal with CONTINUOUS VARIABLES and we write

$$\boxed{P(x' \leq x \leq x' + dx') = f(x')dx'} \Rightarrow \text{by hyp.} \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$f(x') \geq 0$ = probability density function

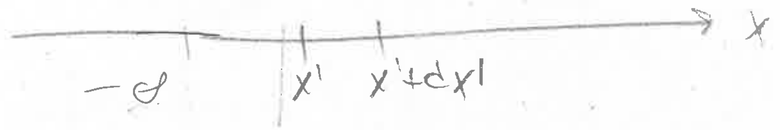
it <u>must</u> be $P(x') = 0$ because :



and $P(x') + P(x'') + \dots = \infty$!
while it must be ≤ 1

$P(S) = 1 \Rightarrow$ if x is continuous, then $P(S)$ must be a <u>linear functional</u> over $\mathbb{R}$

if



then

$$P(-\infty \le x \le x'+dx') - P(-\infty \le x \le x') = P(x' \le x \le x'+dx')$$

So, we def:

$$F(x') = P(-\infty \le x \le x') \Rightarrow$$

$$\Rightarrow f(x)\,dx = F(x+dx) - F(x) = dF(x)$$

and

$$F(x) = \int_{-\infty}^{x} f(x')\,dx'$$

Cumulative distribution function

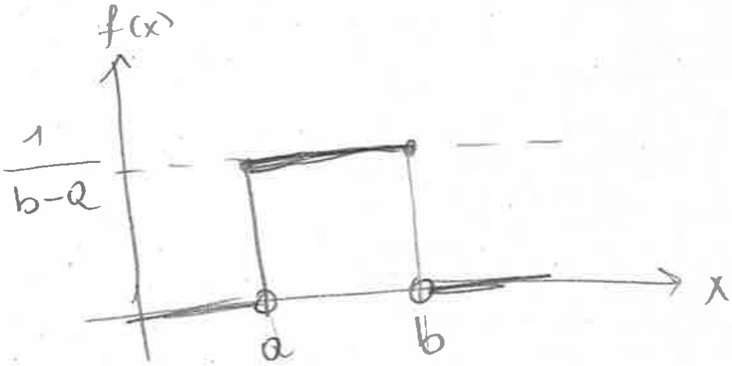
Since, $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow 0 \le F(x) \le 1$,

$F(x)$ is non-decreasing and

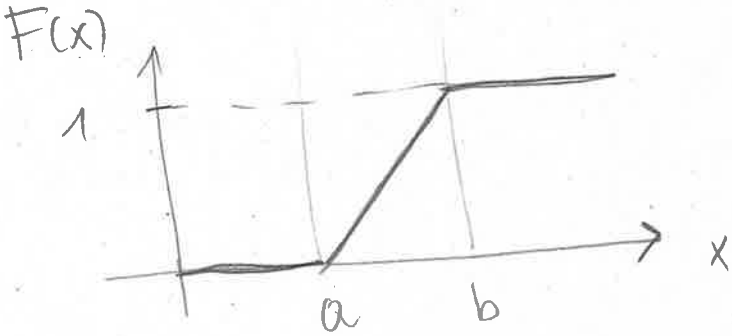
$$P(a < x \le b) = F(b) - F(a)$$

examples

1) Uniform distribution:



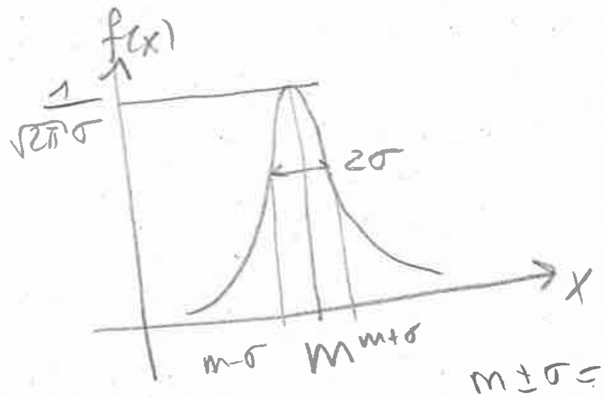
$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & x < a, x > b \end{cases}$$



$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1 & x > b \end{cases}$$

2) Gaussian distribution

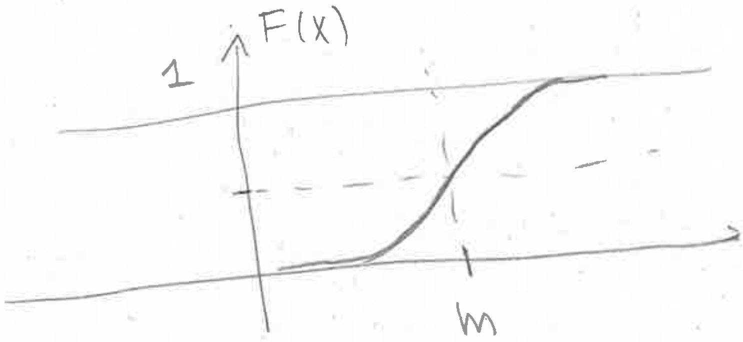
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$$



$m \pm \sigma =$ INFLECTION POINTS

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(x'-m)^2}{2\sigma^2}} dx' = \frac{1}{2}\left(1 + \text{erf}\left[\frac{x-m}{\sqrt{2}\sigma}\right]\right)$$

where $\quad \text{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx$



- Expectation value of any function $u(x)$ is:

$$E[u(x)] = \int_{-\infty}^{\infty} u(x) f(x) dx$$

assuming the integral is finite: By def.

$$E[\alpha u(x) + \beta v(x)] = \alpha E[u(x)] + \beta E[v(x)]$$

The n^{th} moment of a random variable x is:

$$\alpha_n \equiv E[x^n] = \int_{-\infty}^{\infty} x^n f(x) dx$$

and the n^{th} central moment of x (about the mean α_1):

$$m_n \equiv E[(x-\alpha_1)^n] = \int_{-\infty}^{\infty} (x-\alpha_1)^n f(x) dx$$

The most commonly used are MEAN μ and VARIANCE σ^2:

$$\mu \equiv \alpha_1$$
$$\sigma^2 = V[x] \equiv m_2 = \alpha_2 - \mu^2$$

$\sigma = \sqrt{V[x]}$ = STANDARD DEVIATION

NOTE THAT: $V[cx+k] = c^2 V[x]$

Examples:

- Uniform distr:

$$\mu = E[x] = \frac{a+b}{2} \quad ; \quad \sigma^2 = \frac{(b-a)^2}{12}$$

- Gaussian distrib -

$$\mu = m$$

$$\sigma^2 = E[(x-m)^2]$$

Characteristic Function:

~~The~~ $\phi(u)$ associated to pdf $f(x)$ is:

$$\phi(u) = [e^{iux}] = \int_{-\infty}^{\infty} e^{iux} f(x)dx$$

ϕ uniquely det by f and vice versa. It is easy to see that:

$$\boxed{\alpha_n = \int_{-\infty}^{\infty} x^n f(x)dx = \left(\frac{1}{i}\right)^n \frac{d^n\phi}{du^n}\Big|_{u=0}}$$

- 2 random variables =

joint pdf $f(x,y)$

if independent $f(x,y) = f_1(x)\, f_2(y)$

where

$$\left.\begin{aligned} f_1(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\ f_2(y) &= \int_{-\infty}^{\infty} f(x,y)\,dx \end{aligned}\right\} \text{ marginal distributions}$$

The CONDITIONAL pdf of y given fixed x with $f_1(x) \neq 0$ is

$$f_3(y|x) = \frac{f(x,y)}{f_1(x)}$$

and, similarly

$$f_4(x|y) = \frac{f(x,y)}{f_2(x)}$$

From Bayes'

$$f_3(y|x)\, f_1(x) = f_4(x|y)\, f_2(y)$$

- Mean:

$$\mu_x = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} x\, f_1(x)\,dx$$

same for μ_y

- covariance of x and y

$$\mathrm{cov}[x,y] = E[(x-\mu_x)(y-\mu_y)] = E[xy] - \mu_x \mu_y$$
$$= \langle xy\rangle - \langle x\rangle\langle y\rangle$$