* Lecture 8 *

ANDREA AIELLO
SS2017 FAU-ERLANGEN
IK-1

– The energy operator –

In classical mechanics the energy of a particle is:

$$E = \underset{\substack{\uparrow \\ \text{kinetic energy}}}{K} + V \leftarrow \text{potential energy}$$

where $K = \frac{1}{2} m \vec{v}^2 = \frac{\vec{p}^2}{2m}$

$$V = V(\underline{r})$$

$\uparrow$ position vector of the particle

In Quantum Mechanics:

$$\underline{r} \rightarrow \hat{\underline{r}} = (\hat{x}, \hat{y}, \hat{z})$$

$$\underline{p} \rightarrow \hat{\underline{p}} = -i\hbar \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Therefore $K \rightarrow \frac{\hat{\underline{p}}^2}{2m} = -\frac{\hbar^2}{2m} \nabla^2.$

$$V \rightarrow V(\hat{\underline{r}})$$

Now we understand the meaning of the Schrödinger equation for a free particle:

$$i\hbar \frac{\partial}{\partial t} \psi(\underline{r},t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\underline{r},t)$$

$$= \frac{\hat{\underline{p}}^2}{2m} \nabla^2 \psi(\underline{r},t)$$

The energy operator

$$\frac{\hat{\underline{p}}^2}{2m} + V(\hat{\underline{r}}) \equiv \hat{H}$$

is called the HAMILTONIAN of the system and is denoted with the letter $\hat{H}$.

For systems more general than a single particle

$$\psi(\underline{r},t) \to \psi(q,t)$$

but the Sch. eq. remains the same:

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi}$$

where $\hat{H}$ now is the Hamiltonian for the system.

For a <u>single particle</u> $\hat{H}$ has <u>ALWAYS</u> the form:

$$\boxed{\hat{H} = \frac{\hat{\underline{p}}^2}{2m} + V(\underline{r})}$$

For a free-particle

$$\hat{H}_0 = \frac{\hat{\underline{p}}^2}{2m}$$

The "0" denotes free particle

and $[\hat{H}_0, \hat{\underline{p}}] = 0 \Rightarrow \hat{H}$ and $\hat{\underline{p}}$ have the same eigenfunctions. We know that

$$\hat{\underline{p}}\,\phi(\underline{r},\underline{p}') = \underline{p}'\phi(\underline{r},\underline{p}')$$

where

$$\phi(\underline{r},\underline{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \exp\left(\frac{i}{\hbar}\underline{p}\cdot\underline{r}\right)$$

Therefore

$$\begin{aligned}\hat{p}_i^2\,\phi(\underline{r},\underline{p}') &= \hat{p}_i(\hat{p}_i\phi(\underline{r},\underline{p}')) \\ &= \hat{p}_i(p_i'\phi(\underline{r},\underline{p}')) \\ &= p_i'^2\phi(\underline{r},\underline{p}') \quad \Rightarrow\end{aligned}$$

$$\begin{aligned}\Rightarrow \hat{H}_0\phi(\underline{r},\underline{p}') &= \frac{\underline{p}'\cdot\underline{p}'}{2m}\phi(\underline{r},\underline{p}') \\ &= \frac{p_x'^2 + p_y'^2 + p_z'^2}{2m}\phi(\underline{r},\underline{p}')\end{aligned}$$

Homework: Calculate a) $[\hat{H}, \hat{\underline{p}}]$ and d) $[\hat{H}, \hat{\underline{r}}]$ when

$$\hat{H} = \frac{\hat{\underline{p}}^2}{2m} + V(\hat{\underline{r}}) \quad \text{for any } V(\hat{\underline{r}}) \in \mathbb{R}$$

In some cases, when the particle is interacting with another system, it is possible to describe a single part by a TIME-DEPENDENT Hamiltonian

$$\hat{H} = \hat{H}(\underline{r}, t)$$

This is the case, e.g., of an atom interacting with an electromagnetic wave.

For the moment, we consider only TIME-INDEPENDENT Hamiltonians of the form:

$$\hat{H} = \hat{H}(\underline{r})$$

In this case:

$$\boxed{i\hbar \frac{\partial}{\partial t}\psi(\underline{r}, t) = \hat{H}(\underline{r})\psi(\underline{r}, t)} \qquad (1.4)$$

can be solved separating the variables:

$$\psi(\underline{r}, t) = u(\underline{r})T(t) \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} i\hbar \frac{\partial \psi}{\partial t} &= i\hbar u \frac{\partial T}{\partial t} \\ \hat{H}(\underline{r})\psi &= T(t)(\hat{H}u)(\underline{r}) \end{aligned}\right\} \Rightarrow$$

The function $u(\underline{r})$ is usually denoted as: STATIONARY STATE

$\Rightarrow$ (1.4) becomes: $i\hbar\, u(\underline{r}) \frac{\partial T}{\partial t}(t) = T(t)\,(\hat{H}u)(\underline{r})$

We divide both sides by $\psi(\underline{r},t) = u(\underline{r})T(t)$; to obtain:

$$\underbrace{i\hbar \frac{1}{T(t)} \frac{\partial T}{\partial t}(t)}_{\text{function of } t} = \underbrace{\frac{1}{u(\underline{r})}(\hat{H}u)(\underline{r})}_{\text{function of } \underline{r}}$$

So, if

$$f(t) = g(\underline{r})$$

the only possibility is that they are both constant. Let "E" denote this constant:

$$\begin{cases} i\hbar \dfrac{\partial T(t)}{\partial t} = E\, T(t) & (1.5a) \\ \underbrace{\hat{H}(\underline{r})\, u(\underline{r}) = E\, u(\underline{r})} & (1.5b) \end{cases}$$

Two independent equations

$\equiv$ Time-independent Schrödinger equation

It looks like an eigenvalue equation which can be solved (hopefully!) when $\hat{H}(\underline{r})$ is known.

Suppose to know the eigenfunctions $\psi_n(\underline{r})$ and the eigenvalues E_n of $\hat{H}(\underline{r})$:

$$\boxed{\hat{H}\psi_n = E_n \psi_n} \quad (1.6)$$

(we suppose that they are discrete. For a free-particle this is not true).

Then the time evolution of the state n:

$$\psi_n(\underline{r}) \longrightarrow \psi_n(\underline{r},t)$$

is governed by (1.5a) with $E = E_n$:

$$i\hbar \frac{\partial T}{\partial t} = E_n T$$

whose solution is:

$$T(t) = T(0)\exp\left(-\frac{i}{\hbar}E_n t\right)$$

But if $\Psi(\underline{r},0) = u(\underline{r})T(0) = \psi_n(\underline{r}) \Rightarrow T(0) = 1$

and

$$\boxed{\psi_n(\underline{r},t) = \exp\left(-\frac{i}{\hbar}E_n t\right)\psi_n(\underline{r})} \quad (2.6)$$

These eigenvalues and eigenfunctions exists if $\hat{H}$ is Hermitian. Now we show that this is the case.

- Proof that $\hat{H} = \hat{H}^{+}$ -

We start from $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$

and $\int \psi^{*}\psi \, dq = 1 \Rightarrow$

$$\Rightarrow \frac{d}{dt}\left(\int \psi^{*}\psi \, dq\right) = 0$$

$$\Leftrightarrow \int \left(\frac{\partial \psi^{*}}{\partial t}\psi + \psi^{*}\frac{\partial \psi}{\partial t}\right) dq = 0$$

$$\Leftrightarrow \int \left[\left(\frac{1}{\cancel{i\hbar}}\hat{H}\psi\right)^{*}\psi + \psi^{*}\left(\frac{1}{\cancel{i\hbar}}\hat{H}\psi\right)\right] dq = 0$$

$$\Leftrightarrow \int \psi(\hat{H}^{*}\psi^{*})\, dq - \int \psi^{*}(\hat{H}\psi)\, dq = 0$$

$$\Leftrightarrow \int \psi^{*}(\hat{H}^{+}\psi)\, dq - \int \psi^{*}(\hat{H}\psi)\, dq = 0$$

$$\Leftrightarrow \int \psi^{*}(\hat{H}^{+} - \hat{H})\psi \, dq = 0$$ ← This must be true for <u>any</u> $\psi \Rightarrow$

$$\Rightarrow \hat{H}^{+} = \hat{H}$$

<u>c.v.d</u>

(8-8)

Now, since $\hat{H} = \hat{H}^{+}$ the eigenfunctions are orthogonal and complete. This implies that at $t=0$ we can write:

$$\psi(q,0) = \sum_n a_n \psi_n(q)$$

where $$a_n = \int \psi_n^{*}(q)\psi(q,0)\,dq$$

Since for $t>0$ $\psi_n(q) \to \psi_n(q)\exp\left(-\frac{i}{\hbar}E_n t\right) \Rightarrow$

$$\Rightarrow \boxed{\psi(q,t) = \sum_n a_n \psi_n(q) \exp\left(-\frac{i}{\hbar}E_n t\right)} \quad (1.8)$$

— Theorem —

If a physical quantity f is CONSERVED, that is its value does not change with time, the corresponding operator $\hat{f}$ commute with $\hat{H}$.

— Proof —

Let $\hat{f}$ such that has mean value $\langle \hat{f} \rangle_{\psi}$ when the system is prepared in the state ψ, and let $\frac{d\hat{f}}{dt} \equiv \dot{\hat{f}}$. Then $\dot{\hat{f}}$ is DEFINED the time-derivative of such operator as the quantity such that:

$$\boxed{\langle \dot{\hat{f}} \rangle_{\psi} \equiv \frac{d}{dt}\langle \hat{f} \rangle_{\psi}} \quad \forall \psi \quad (2.8)$$

$\Leftarrow$ This is a definition

By definition:

$$\frac{d}{dt}\langle\hat{f}\rangle_\psi = \frac{d}{dt}\int \psi^*(q,t)\,\hat{f}(t)\,\psi(q,t)\,dq$$

$$= \int \frac{\partial\psi^*}{\partial t}(\hat{f}\psi)\,dq + \int \psi^*\left(\hat{f}\frac{\partial\psi}{\partial t}\right)dq + \int \psi^*\left(\frac{\partial\hat{f}}{\partial t}\psi\right)dq$$

but $\frac{\partial\psi}{\partial t} = \frac{1}{i\hbar}\hat{H}\psi \Rightarrow$

$$\Rightarrow \frac{d}{dt}\langle\hat{f}\rangle_\psi = \frac{-1}{i\hbar}\int (\hat{H}^*\psi^*)(\hat{f}\psi)\,dq$$

$$+ \frac{1}{i\hbar}\int \psi^*(\hat{f}\hat{H}\psi)\,dq + \int \psi^*\frac{\partial\hat{f}}{\partial t}\psi\,dq$$

$$= -\frac{1}{i\hbar}\int\left[\psi^*(\underbrace{\hat{H}^+}_{=\hat{H}}\hat{f}\psi) - (\hat{f}\hat{H}\psi)\right]dq + \int \psi^*\frac{\partial\hat{f}}{\partial t}\psi\,dq$$

$$= \int \psi^*\left\{\frac{1}{i\hbar}[\hat{f},\hat{H}] + \frac{\partial\hat{f}}{\partial t}\right\}\psi\,dq \qquad (1.9)$$

On the other hand, by definition:

$$\langle\dot{\hat{f}}\rangle_\psi = \int \psi^*\frac{d\hat{f}}{dt}\psi\,dq \qquad (2.9)$$

Imposing (1.9) = (2.9) we find: (to be valid $\forall\psi$)

$$\boxed{\frac{d\hat{f}}{dt} = \frac{1}{i\hbar}[\hat{f},\hat{H}] + \frac{\partial\hat{f}}{\partial t}} \qquad (3.9)$$

The last term

$$\frac{\partial \hat{f}}{\partial t}$$

is non-zero only if $\hat{f}$ has an explicit time-dependence. If this is not the case, as in the conserved observables, we have:

$$\frac{d\hat{f}}{dt} = \frac{i}{\hbar}[\hat{H},\hat{f}] = 0 \quad \Rightarrow \quad [\hat{H},\hat{f}] = 0 \qquad \text{c.v.d}$$

If $\frac{\partial \hat{f}}{\partial t} = 0$, then its mean value with respect to the energy eigenfunctions Ψ_n is time independent because:

$$\frac{d}{dt}\int \Psi_n^*(q,t)\,\hat{f}\,\Psi_n(q,t)\,dq$$

$$= \frac{d}{dt}\int \Psi_n^*(q)\, e^{\frac{i}{\hbar}E_n t}\,\hat{f}\,\Psi_n(q)\, e^{-\frac{i}{\hbar}E_n t}\,dq$$

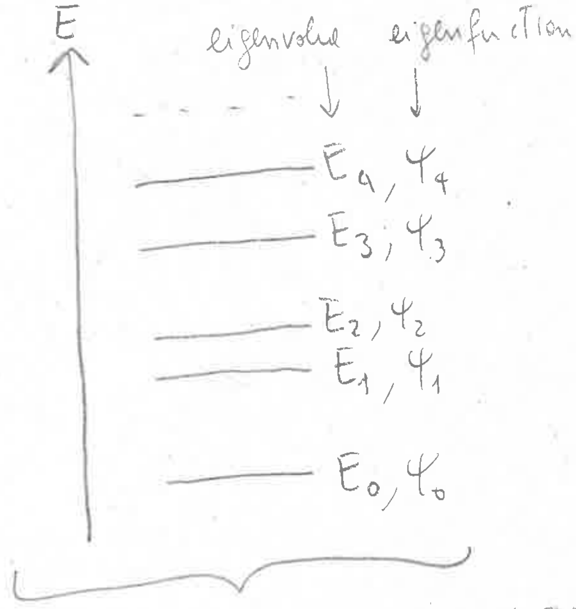
Their product = 1

$$= \frac{d}{dt}\int \Psi_n^*(q)\,\hat{f}\,\Psi_n(q)\,dq = 0$$

Jargon:

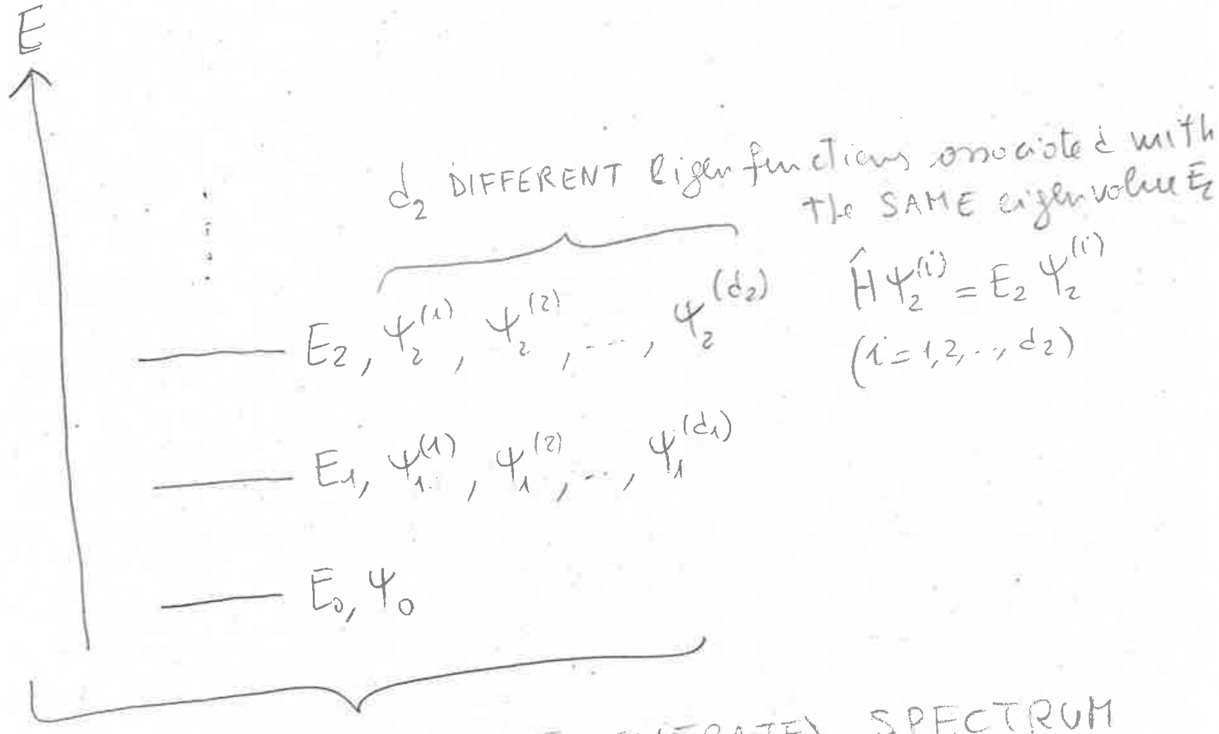
SPECTRUM of $\hat{H}$ = set of all eigenvalues of $\hat{H}$

A typical DISCRETE spectrum looks like:



SIMPLE (OR NONDEGENERATE) SPECTRUM

In the case of degenerate eigenvalues we have:



NON-SIMPLE (OR DEGENERATE) SPECTRUM

When does a degenerate spectrum occur?

When there are two (or more) CONSERVED observables, say f and g, that are INCOMPATIBLE (that is, their associated Hermitian operators $\hat{f}$ and $\hat{g}$ do not commute: $[\hat{f}, \hat{g}] \neq 0$, as, e.g., $[\hat{x}, \hat{p}] = i\hbar$), but commute with $\hat{H}$ (because, by hypothesis, they are conserved):

$$\hat{f}, \hat{g} : [\hat{f}, \hat{g}] \neq 0 \quad \text{and} \quad [\hat{f}, \hat{H}] = 0 = [\hat{g}, \hat{H}]$$

Let $\hat{g} u_n = g_n u_n$ and $\hat{f} v_m = f_m v_m$ and $\hat{H}\psi = E\psi$

Choose ψ such that $\hat{f}\psi = f\psi$, so f has a determinate value. I can do because $[\hat{H}, \hat{f}] = 0$;

$$\hat{H}(\hat{f}\psi) = \hat{f}(\hat{H}\psi) = E(\hat{f}\psi)$$

So $\hat{f}\psi$ is an $\underbrace{\text{eigenstate}}^{\text{synonimous of eigenfunction}}$ of $\hat{H}$ with eigenvalue E

However, also $\hat{H}(\hat{g}\psi) = \hat{g}(\hat{H}\psi) = E(\hat{g}\psi)$ is an eigenstate of $\hat{H}$ associated with the SAME eigenvalue E.

So, if $\hat{f}\psi \neq \underset{\substack{\uparrow \\ \text{c-number}}}{\lambda} \hat{g}\psi$, then $\hat{H}$ has at least 2 eigenstates

ψ and $\hat{g}\psi$ ($\hat{f}\psi = f_n\psi \propto \psi$ by hypotesis) associated with the same eigenvalue E

However, if it were

$$\hat{f}\psi = \lambda \hat{g}\psi \qquad \text{and } \hat{f}\psi = f\psi$$

Then we would have $\quad f\psi = \lambda(\hat{g}\psi) \Leftrightarrow$

$$\Leftrightarrow \quad \hat{g}\psi = \left(\frac{f}{\lambda}\right)\psi \equiv g\psi$$

So ψ would also be eigenstate of $\hat{g}$. But this cannot be true <u>in general</u> because $[\hat{f}, \hat{g}] \neq 0$.

Note: There can exist common eigenstates of noncommuting operators. These eigenstates span a subspace where $[\hat{f}, \hat{g}] = 0$. In this subspace you can measure simultaneously $\hat{f}$ and $\hat{g}$ with a determined value.

* Example * let $F = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$; $G = \underbrace{\begin{bmatrix} 0 & x & 0 \\ x & 0 & 0 \\ 0 & 0 & y \end{bmatrix}}_{\text{eigenvalues } = (-x, x, y)}$

Then $[F, G] = \begin{bmatrix} 0 & (a-b)x & 0 \\ -(a-b)x & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$

$[F,G] = 0$ in this 1-dimensional subspace

The eigenvectors of F and G are, respectively,

$\underline{v}_n = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$; $\underline{u}_n = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ so $\underline{v}_3 = \underline{u}_3$

So, if $\hat{f}$ and $\hat{g}$ have some non-common eigenstates (and it must be so because $[\hat{f},\hat{g}]\neq 0$) then $\hat{H}$ has degenerate eigenvalues.

Let $\{\psi_n^{(1)}, \psi_n^{(2)}, \dots, \psi_n^{(d_n)}\}$ a set of d_n eigenfunctions associated with the same eigenvalue E_n. Then, ANY linear combination

$$\psi = \sum_{i=1}^{d_n} c_i \psi_n^{(i)}$$

is associated with E_n because

$$\hat{H}\psi = \sum_i c_i \hat{H}\psi_n^{(i)} = E_n \sum_i c_i \psi_n^{(i)}$$

So, I can always choose d_n linear combinations

$$\phi_{ni} = \sum_{j=1}^{d_n} c_{ij}\psi_n^{(j)} \qquad (i=1,2,\dots,d_n)$$

such that:

$$\int \phi_{ni}^* \phi_{nj}\, dq = \delta_{ij}$$

HOMEWORK – Prove that there are infinitely many different sets $\{\phi_{n1}, \phi_{n2}, \dots, \phi_{nd_n}\}$ of orthogonal wavefunctions associated to E_n by showing that $C > D$, where

C = number of independent coefficients of the linear transformation generating the set $\{\phi_{n1}, \dots, \phi_{nd_n}\}$ from the set $\{\psi_n^{(1)}, \dots, \psi_n^{(d_n)}\}$

D = number of conditions of normalization and orthogonality satisfied by the set $\{\phi_{n1}, \dots, \phi_{nd_n}\}$

* Lecture 9 *

ANDREA AIELLO
SS 2017 FAU-ERLANGEN
IK-1

– Matrices and unitary operators –

Let $\hat{H}\psi_n = E_n \psi_n$ and ψ arbitrary

Then $\psi(t) = \sum_n a_n(t)\psi_n$. Let f be an observable.

$$a_n(t) = a_n(0)\, e^{-\frac{i}{\hbar}E_n t} \equiv a_n(0)\, e^{-i\omega_n t}$$

$$\omega_n \equiv \frac{E_n}{\hbar}$$

$$\langle \hat{f} \rangle_\psi \underset{\text{def.}}{\equiv} \int \psi^* (\hat{f}\psi)\, dq \;;$$

$$= \int \Big(\sum_n a_n \psi_n\Big)^* \hat{f} \Big(\sum_m a_m \psi_m\Big) dq$$

$$= \sum_{n,m} a_n^* a_m \underbrace{\int \psi_n^* (\hat{f}\psi_m)\, dq}_{\equiv f_{nm}}$$

← matrix element

$$= \sum_{n,m} a_n^*(t) f_{nm} a_m(t)$$

$$= \sum_{n,m} a_n^*(0) f_{nm} a_m(0)\, e^{i\omega_{nm} t}$$

$$\omega_{nm} \equiv \frac{E_n - E_m}{\hbar}$$

≡ Transition frequency between level n and m

By definition, the operator $\dot{\hat{f}}$ is such that:

$$\langle \dot{\hat{f}} \rangle_\psi = \frac{d}{dt} \langle \hat{f} \rangle_\psi$$

$$= \frac{d}{dt} \sum_{n,m} a_n^*(0) f_{nm} a_m(0)\, e^{i\omega_{nm} t}$$

$$= i \sum_{n,m} \omega_{nm}\, a_n^*(0) f_{nm}\, a_m(0)\, e^{i\omega_{nm} t}$$

However, by def.

$$\langle \dot{\hat{f}} \rangle_\psi = \sum_n a_n^*(0)\, \dot{f}_{nm}\, a_m(0)\, e^{i\omega_{nm}t} \Rightarrow$$

$$\Rightarrow \boxed{\dot{f}_{nm} = i\,\omega_{nm} f_{nm}}$$

It is easy to see that for a generic operator $\hat{f}$:

$$(\hat{f}^+)_{nm} = (f_{mn})^*$$

because

$$(\hat{f}^+)_{nm} \equiv \int \psi_n^* (\hat{f}^+ \psi_m)\, dq$$

$$= \int \psi_m (\hat{f}^* \psi_n^*)\, dq = \left(\int \psi_m^* (\hat{f}\psi_n)\, dq\right)^*$$

$$= (f_{mn})^*$$

The Schrodinger equation

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi} \qquad (1,9)$$

can also be written in matrix form

Let $\hat{f}$: $\hat{f}\phi_n = f_n \phi_n \Rightarrow \psi(q,t) = \sum_n a_n(t)\,\phi_n(q)$

Multiply 1.9 by $\phi_n^*(q)$ and integrate

$$i\hbar \underbrace{\frac{d}{dt}\int \phi_n^*(q)\,\psi\, dq}_{= a_n(t) \text{ by def}} = \underbrace{\int \phi_n^*(q)\,\hat{H}\,\psi\, dq}$$

$$= \int \phi_n^* \hat{H} \left(\sum_m a_m(t)\,\phi_m\right) dq$$

$$= \sum_m a_m(t)\, H_{nm} \quad ; \; H_{nm} \equiv \int \phi_n^* \hat{H}\, \phi_m\, dq$$

Therefore we get:

$$\boxed{i\hbar \frac{d}{dt} a_n(t) = \sum_m H_{nm}\, a_m(t)} \qquad (1.3)$$

If $\hat{H}$ does not depend on t (isolated system)

this is a first-order differential linear system whose solution is trivial for finite dimension N:

Let $H = [H_{nm}]$ a $N \times N$

and U a unitary matrix such that: (U is time-indep.)

$$UHU^{+} = D = \mathrm{diag}(E_1, E_2, \ldots, E_N)$$

If $|a\rangle = (a_1, a_2, \ldots, a_N)$ we rewrite (1.3) as:

$i\hbar \frac{d}{dt}|a\rangle = H|a\rangle$ multiply from left by U:

$$i\hbar \frac{d}{dt}(U|a\rangle) = UH|a\rangle$$

identity matrix

$$= UHI|a\rangle$$

$$= (UHU^{+})\,U|a\rangle \qquad \text{Let } |b\rangle \equiv U|a\rangle \Rightarrow$$

$$\Rightarrow \boxed{i\hbar \frac{d|b\rangle}{dt} = D|b\rangle} \Leftrightarrow i\hbar \frac{db_n}{dt} = \lambda_n b_n \Rightarrow b_n(t) = b_n(0)\, e^{-\frac{i}{\hbar}\lambda_n t}$$

Or, equivalently:

$$|b(t)\rangle = e^{-iD\frac{t}{\hbar}}|b(0)\rangle$$

but $|a(t)\rangle = U^{\dagger}|b(t)\rangle \Rightarrow$

$$\Rightarrow \underbrace{U^{\dagger}|b(t)\rangle}_{=|a(t)\rangle} = \underbrace{\left(U^{\dagger} e^{-iD\frac{t}{\hbar}} U\right)}_{=\exp\left(-iH\frac{t}{\hbar}\right)} \underbrace{U^{\dagger}|b(0)\rangle}_{\equiv|a_0\rangle}$$

because $U^{\dagger} e^{-iD\frac{t}{\hbar}} U = U^{\dagger} \sum_k \left(-\frac{it}{\hbar}\right)^k \frac{D^k}{k!} U$

$$= \sum_k \left(-\frac{it}{\hbar}\right)^k \frac{(U^{\dagger}DU)^k}{k!} = e^{-iH\frac{t}{\hbar}}$$

Therefore: $\boxed{|a(t)\rangle = \exp\left(-iH\frac{t}{\hbar}\right)|a_0\rangle}$

There are some system for which is not important to know the spatial dependence of the state,